A CHARACTERIZATION OF THE HALL PLANES
BY PLANAR AND NONPLANAR INVOLUTIONS

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ABSTRACT. In this article, the Hall planes of even order \( q^2 \) are characterized as translation planes of even order \( q^2 \) admitting a Baer group of order \( q \) and at least \( q+1 \) nontrivial elations.

KEY WORDS AND PHRASES. Translation plane, Baer groups, elations.

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1. INTRODUCTION AND BACKGROUND.

Let \( \Sigma \) denote an affine Desarguesian plane of order \( q^2 \) coordinatized by a field \( F \) isomorphic to \( GF(q^2) \). Let \( \mathcal{N} \) denote the net defined on the points of \( \Sigma \) whose lines have slopes in \( GF(q) \cup \{\infty\} \). Let \( \sigma \) denote the involution defined by \( (x,y) \rightarrow (x^q,y^q) \) where \( x,y \in F \). Let \( \hat{\mathcal{P}}^* \) denote the kernel homology group of \( \Sigma \) defined by \( (x,y) \rightarrow (ax,ay) \) where \( |a| = q+1, \ a \in F \).

Now derive \( \mathcal{N} \) to obtain the Hall plane \( \Sigma \) of order \( q^2 \). Then the involutions in \( \langle \sigma \rangle \hat{\mathcal{P}}^* \) are central collineations in \( \Sigma \).

If \( \mathcal{E} \) denotes an elation group fixing \( a = (0,0) \) with axis \( \mathcal{L} \) in \( \mathcal{N} \) which acts regularly on the remaining lines of \( \mathcal{N} \) incident with \( a \) then \( \mathcal{E} \) becomes a collineation group of \( \Sigma \) of order \( q \) which fixes a Baer subplane pointwise.

In [3] and [4], Foulser and Johnson classify the translation planes of order \( q^2 \) that admit \( SL(2,q) \). In particular, if \( q^2 > 16 \), the Hall planes are precisely the translation planes admitting \( SL(2,q) \) where the Sylow \( p \)-subgroups for \( q = p^r \) fix Baer subplanes pointwise.

So, the Hall planes of order \( q^2 \) admit a Baer group of order \( q \) and at least \( 1+q \) involutory central collineations.

In this article, we consider translation planes of order \( q^2 \) that admit a Baer group of order \( q \) and \( \geq 1+q \) involutory central collineations. For \( q \) odd, it turns out that there are other (i.e. non Hall) translation planes possessing this configuration of groups. For example, the translation planes \( \pi \) corresponding to the Fisher flock of a quadratic cone in \( PG(3,q) \) for
q \equiv 3 \mod 4$ derive planes $\pi$ admitting such groups (see [5]).

However, for $q$ even, we are able to characterize the Hall planes using these planar and non planar involutions.

Our main result is

**THEOREM A.** Let $\pi$ be a translation plane of even order $q^2$ which admits a Baer collineation group $\mathcal{B}$ of order $q$ and at least $1+q$ nontrivial elations (all groups are assumed to be in the translation complement). Then $\pi$ is the Hall plane of order $q^2$ and conversely, the Hall plane admits such groups.

The proof of theorem A will be given as a series of lemmas. As a preliminary to the proof, we remind the reader of some results required in the arguments.

**RESULT (JHA, JOHNSON [7] (4.1)).** Let $\pi$ be a translation plane of even order $q^2 \neq 64$. Assume $\pi$ admits a Baer group of order $q$ and a dihedral group of order $2(1+q)$ which is generated by elations with affine axes. Then $\pi$ is derivable where the elation axes define a derivable partial spread.

**RESULT II (FOULSER [2] THEOREM 2 AND COROLLARY 3 (2)).** Let $\pi$ be a translation plane of order $q^2$ that admits a Baer group $\mathcal{B}$ of order $q$. (1) Then the Baer subplane $\pi_0 = \text{Fix} \mathcal{B}$ pointwise fixed by $\mathcal{B}$ is Desarguesian. (2) Furthermore, if the collineation group $\mathcal{G} [\pi_0]$ fixing $\pi_0$ pointwise has order $> q$ then the net $\mathcal{N}$ defined by the lines of $\pi_0$ is a derivable net. (3) In the general case, $\mathcal{G} [\pi_0]$ is a subgroup of $\text{AG}(1,q)$, the 1-dimensional affine group over $\text{GF}(q)$.

**RESULT III (JHA, JOHNSON [7]).** Let $\pi$ be a translation plane of even order $q$ that admits a Baer $2$-group of order $\geq 2\sqrt{q}$. Then an elation group with fixed affine axis has order $\leq 2$.

**RESULT IV (A MODIFIED VERSION OF THE MAIN RESULTS OF HERING [6], OSTROM [10]).** Let $\pi$ be a translation plane of even order. Let $\mathcal{G}$ denote the collineation group generated by all elations in the translation complement. If $\mathcal{G}$ is solvable then either $\mathcal{G}$ is an elementary abelian $2$-group or has order $2 \cdot t$ where $t$ is odd.

**RESULT V (JHA-JOHNSON [8]).** Let $\pi$ be a translation plane of even order $q^2$ which admits collineation groups $\mathcal{B}_1, \mathcal{B}_2$ of orders $\geq 2\sqrt{q}$ such that $\mathcal{B}_i$ fixes a Baer subplane $\pi_i$, $i = 1, 2$ pointwise. If $\pi_1 \neq \pi_2$ then $\pi$ is Hall or a known plane of order 16.

2. THE CHARACTERIZATION.

Assume for this section, the assumptions of Theorem A and assume $\pi$ is not Hall.

(2.1) **LEMMA.** Result I is valid for $q^2 = 64$. 
PROOF. \( \pi \) is a translation plane of order 64 that admits a Baer group \( \mathcal{B} \) of order 8 and \( \geq 1+8 \) affine elations. If \( \pi \) is not Hall then \( \mathcal{D} \) still becomes dihedral of order 2 \( \cdot 9 \) and centralizes \( \mathcal{B} \). Let \( \mathcal{C} \) denote the cyclic stem of \( \mathcal{D} \). Let \( \mathcal{C} = \langle g \rangle \). There are \( 8 \cdot 7 \) components of \( \pi \) not in \( \mathcal{N} \) so that \( g \) must fix at least two of these components \( \mathcal{L}_1, \mathcal{L}_2 \). Now \( g \) leaves invariant \( \pi_0 = \text{Fix } \mathcal{B}, \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Thus, \( g \) fixes \( \geq 3 \) mutually disjoint 2m-spaces (if \( q = 2^m \)) over \( GF(2) \). Now the argument given by Jha–Johnson [7] for result I will be valid for \( q^2 = 64 \). This proves (2.1).

Now assume the order of the plane is 16. The translation planes of order 16 are either semifield planes or derived from semifield planes (see Johnson [9] and Dempwolff and Riefart [1]). In any case, the non Hall planes admitting Baer groups of order 4 do not admit \( \geq 5 \) elations.

So, we may assume \( q \neq 4 \).

(2.2) LEMMA. Let \( \mathcal{D} \) denote the collineation group generated by the affine elations. Then \( \mathcal{D} \) is dihedral of order \( 2(q+1) \), acts faithfully on \( \pi_0 \) and centralizes \( \mathcal{B} \).

PROOF. By result IV, no two of the elations can have a common axis. Hence, it follows that \( \mathcal{D} \) is solvable by result IV, \( |\mathcal{D}| = 2 \cdot t \) where \( t \) is odd.

By result II, \( \mathcal{D} \) must normalize \( \mathcal{B} \). Clearly, the elations must have axes nontrivially intersecting \( \pi_0 \) and leaving \( \pi_0 \) invariant. Since a central collineation is uniquely determined by its axis (co axis) and one specified nontrivial image point, it follows that \( \mathcal{D} \) centralizes \( \mathcal{B} \). Hence, if \( y \in \mathcal{D} \cap \mathcal{B} \setminus \langle 1 \rangle \) then the Sylow 2-subgroups of \( \mathcal{D} \) would have order \( \geq 4 \). So \( \mathcal{D} \cap \mathcal{B} = \langle 1 \rangle \).

If \( 1 \neq h \in \mathcal{D} \) fixes \( \pi_0 \) pointwise then the collineation fixing \( \pi_0 \) pointwise has order \( > q \) so that by result II(2), the net \( \mathcal{N} \) (see notation in II(2)) is derivable.

Let \( \pi_1 \) be a Baer subplane of \( \mathcal{N} \) incident with the zero vector \( \alpha \). The infinite points of \( \pi_1 \) are exactly those of \( \pi_0 \). If \( \sigma \) is any elation in \( \mathcal{D} \) then the axis of \( \sigma \) is in \( \pi_1 \) and \( \sigma \) permutes the infinite points of \( \pi_1 \). Hence, \( \sigma \) leaves \( \pi_1 \) invariant and since \( \mathcal{D} \) is generated by elations, it follows that \( \mathcal{D} \) must fix each of the \( q+1 \) Baer subplanes of \( \mathcal{N} \) incident with \( \alpha \). However, this means that \( h \) cannot fix \( \pi_0 \) pointwise.

Thus, \( \mathcal{D} \) acts faithfully on \( \pi_0 \). Now \( \pi_0 \) is Desarguesian by result II(1) and since \( \mathcal{D} \) is generated by elations of \( \pi_0 \), \( \mathcal{D} \leq SL(2,q) \cong PSL(2,q) \) and \( |\mathcal{D}| = 2 \cdot t \) where \( t \) is odd. Thus, \( \mathcal{D} \) is dihedral and admits \( \geq 1+q \) involutions. This proves (2.1).

(2.3) LEMMA. \( \pi \) is derivable with derivable net \( \mathcal{N} \) (in the above notation).

PROOF. (2.2 and result I).

(2.4) LEMMA. Let \( \sigma \) be any elation in \( \mathcal{D} \). Then for any \( \tau \in \mathcal{B} \setminus \langle 1 \rangle \), \( \tau \sigma \) is a Baer involution. Furthermore, if \( \rho \in \mathcal{B} \setminus \langle 1 \rangle \), \( \rho \neq \tau \) then the set of components of \( \pi \) not in \( \mathcal{N} \) fixed by \( \rho \sigma \) is disjoint from the set of components not in \( \mathcal{N} \) fixed by \( \tau \sigma \).

PROOF. If \( \tau \sigma \) is an elation then \( (\tau \sigma)\sigma \in \mathcal{D} \). But \( \mathcal{D} \cap \mathcal{B} = \langle 1 \rangle \). Hence, \( \tau \sigma \) is a Baer involution.

Let \( \mathcal{L} \) be a component fixed by both \( \tau \sigma \) and \( \rho \sigma \). Then \( (\tau \sigma)(\rho \sigma) \) also fixes \( \mathcal{L} \) and
(\tau \sigma \rho \sigma = \tau \rho \sigma^2 = \tau \rho \in D \text{ fixes } L. \text{ Thus } L \text{ is a component of } N.

(2.5) LEMMA. Let \sigma be any elation in D. Then each component of \pi not in N is fixed by exactly one Baer involution in \sigma(D - \langle 1 \rangle).

PROOF. By (2.4), there are q(q-1) distinct components fixed by some involution in \sigma(D - \langle 1 \rangle). Since there are exactly q(q-1) components not in N, (2.5) is proved.

(2.6) LEMMA. Let D = \langle \sigma, \chi \rangle where \sigma, \chi are distinct elations. Each component L of \pi not in N is fixed by \sigma \chi.

PROOF. By (2.5), there exists a Baer involution \rho \sigma \in (D - \langle 1 \rangle) \sigma which fixes L and similarly, there is a Baer involution \tau \chi in (D - \langle 1 \rangle) \chi which fixes L.

Thus (\rho \sigma)(\tau \chi) also fixes L. However, (\rho \sigma)(\tau \chi) = (\rho \sigma)(\sigma \chi) by (2.1). Further, 
((\rho \sigma)(\sigma \chi))^2 = (\rho \tau)^2(\sigma \chi)^2 \text{ again by } (2.1) = (\sigma \chi)^2. \text{ Since } |\langle \sigma \chi \rangle| = q+1 \text{ and } q+1 \text{ is odd, then } (\sigma \chi)^2 = (\sigma \chi)^2. \text{ Thus, } (\sigma \chi)^2 \text{ and thus } \sigma \chi \text{ fixes } L.

(2.7) LEMMA. Let \bar{\pi} denote the translation plane obtained from \pi by deriving N. Then \bar{D} is a collineation group of \bar{\pi}.

PROOF. \bar{D} leaves N invariant.

(2.8) LEMMA. Let \mathcal{C} denote the cyclic stem of D of order q+1. Then \mathcal{C} is a kernel homology group of \bar{\pi}.

PROOF. It was noted in the proof to (2.1) that D must fix each Baer subplane incident with \alpha in N. Hence, the stem \mathcal{C} of D must fix each such Baer subplane. The components of \bar{\pi} are the components of \pi not on N and the Baer subplanes of N which are incident with \alpha. By (2.6), if D = \langle \sigma, \chi \rangle then \mathcal{C} = \langle \sigma \chi \rangle so that \mathcal{C} fixes each component of \pi not in N. Thus, \mathcal{C} must induce a kernel homology group in \bar{\pi}.

Let the kernel of \bar{\pi} be isomorphic to GF(2^r) \leq GF(q^2). Let q = 2^m so that r|2m. then 1+q \mid 2^r - 1 \text{ by } (2.7). \text{ Thus, } r > m \text{ so that } r = 2m.

Thus, the kernel of \bar{\pi} is isomorphic to GF(q^2) so that \bar{\pi} is Desarguesian. Thus, \pi must be Hall and we obtain the proof to theorem A.

REFERENCES


