SUPERSETS FOR THE SPECTRUM OF ELEMENTS IN EXTENDED BANACH ALGEBRAS

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ABSTRACT. If $A$ is a Banach Algebra with or without an identity, $A$ can be always extended to a Banach algebra $\overline{A}$ with identity, where $\overline{A}$ is simply the direct sum of $A$ and $\mathbb{C}$, the algebra of complex numbers. In this note we find supersets for the spectrum of elements of $\overline{A}$.

KEY WORDS AND PHRASES. Banach Algebra, spectrum of elements and quasi-singular elements.

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1. INTRODUCTION.

Let $A$ be a Banach algebra. Then we know that the set $\overline{A} = \{(x,\alpha) : x \in A, \alpha \text{ complex}\}$ together with the operations $(x,\alpha) + (y,\beta) = (x + y, \alpha + \beta)$ and $(x,\alpha) (y,\beta) = (xy + \beta x + \alpha y, \alpha \beta)$, and norm $\| (x,\alpha) \| = \|x\| + |\alpha|$ is a Banach algebra, whose identity element is $(0,1)$. Although this is usually done for algebras $A$ without identity, to extend them to algebras with identity; we can also start with a Banach algebra $A$ with identity (in this case the identity of $A$ is no more an identity for $\overline{A}$).

2. MAIN RESULTS.

DEFINITION 2.1. An element $x$ in a Banach algebra $A$ is called quasi-regular if $x\circ y = y\circ x = 0$ for some $y \in A$, where $x\circ y = x + y - xy$. $x\circ y$ is called the circle operation. $x$ is called quasi-singular if it is not quasi-regular. For an element $x$ in $A$, the special radius of $x$ is defined by

$$r(x) = \lim_{n \to \infty} \| x^n \|^{1/n}.$$

THEOREM 2.1. Let $\overline{A}$ be the extension of $A$, as above and let $\delta_{\overline{A}}((x,\alpha))$ denote the spectrum of $(x,\alpha)$ in $\overline{A}$, then

$$\delta_{\overline{A}}((x,\alpha)) \subseteq \{\alpha\} \cup \delta_A(x + \alpha),$$

if $A$ already has an identity, and

$$\delta_{\overline{A}}((x,\alpha)) \subseteq \{\lambda : |\lambda - \alpha| \leq r(x)\}.$$
if A does not have an identity.

**PROOF.** First suppose A has an identity. Let $\lambda$ be a complex number, then 
\[(x, a) - (0, \lambda) = (x, a - \lambda).\] If $\lambda \neq a$, then
\[(x, a - \lambda) (y, \frac{1}{a - \lambda}) = (xy + \frac{1}{a - \lambda} x + (a - \lambda) y, 1).\]

Now, if $\lambda \neq a$ and $\frac{1}{a} \notin \delta_A (x + a)$, then the equation
\[xy + \frac{1}{a - \lambda} x + (a - \lambda) y = 0 \quad (2.1)\]
has a solution. To see this, write (2.1) as $(a - \lambda) xy + x + (a - \lambda)^2 y = 0$ or
\[(a - \lambda)[x + a - \lambda] y = -x \text{ or } y = \frac{1}{a - \lambda} (x + a - \lambda)^{-1} (-x). \quad (x + a - \lambda)^{-1} \text{ exists since } \lambda \notin \delta_A (x + a).\]
This implies $\{a\} \cap \{-\delta_A (x + a)\} \subseteq -\delta_A ((x, a))$, and, therefore, we have:
\[\delta_A ((x, a)) \subseteq \{a\} \cup \delta_A (x + a).\]

Now suppose A does not have an identity and let $\lambda \neq a$. If $\frac{1}{a - \lambda} x$ is quasi-irregular, then there exists an element $z$ in A such that:
\[\frac{1}{a - \lambda} xz + \frac{1}{a - \lambda} x + z = 0.\]
If we take $y = \frac{1}{a - \lambda} z$, then we have:
\[xy + \frac{1}{a - \lambda} x + (a - \lambda) y = 0.\]

Hence, $\{a\} \cap \{\lambda | \lambda \neq a: \frac{1}{a - \lambda} x \text{ is quasi-singular}\} \subseteq -\delta_A ((x, a))$. For an element $a$ in a Banach algebra, the inequality $r(a) < 1$ implies a is a quasi-regular with quasi-inverse $a = \sum_{n=1}^{\infty} a_n^n$ (Rickart [1]). Hence, for an element $\frac{1}{a - \lambda} x$ to be quasi-singular, it is necessary to have $r(\frac{1}{a - \lambda} x) \geq 1$; that is $r(x) \geq |x - a|$

**REFERENCES**