ABSTRACT. In this paper we will deal with upper and lower bounds for \( \pi(x + y) - \pi(x) \). In fact, given \( q \) with \( 0 < q \leq 1 \), for sufficiently large integers \( m, n \) such that \( m > n \geq qm > 2 \) we show that 
\[
\pi(m + n) - \pi(m) < \ln(n)\pi(n)/\ln(m + 1).
\]
Moreover, explicit bounds are obtained and a wider range is given under the assumption of the Riemann hypothesis. Let \( m, n \) be positive integers with \( m > 2657 \). Let \( 1 \leq \theta < 2 \) and \( m \geq n \geq m^{1/\theta} \). If the Riemann hypothesis holds, then
\[
\pi(m + n) - \pi(m) < n/\ln(m + 1) + \sqrt{n} + n \ln(n^\theta + n)/4.\pi.
\]
(Here \( \pi(x) \) the number of primes \( \leq x \).)

KEY WORDS AND PHRASES. Primes. Small intervals. \( \pi(x + y) \leq \pi(x) + \pi(y) \).

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1. INTRODUCTION.

There are several accounts dealing with the validity of the conjecture that for \( x > 1 \) and \( y > 1 \),
\[
\pi(x + y) \leq \pi(x) + \pi(y).
\]
(1.1)

For example [1], [2], [3] deal with (1.1), whereas in [4] there is a discussion of the conjecture of the following form:
\[
\pi(x + y) < \pi(x) + \pi(y) + cy/\ln^2(y).
\]
(1.2)

(Here we let \( x \geq y \geq 1 \) and \( c > 0 \).) In fact, one of the two authors of [4] believes that (1.2) is true, whereas the other one does not.

What is interesting to this author is a paper written by Hensley and Richards [5]; they proved that if the prime k-tuple conjecture is true then (1.1) is false. Furthermore, assuming that the k-tuple conjecture is true they have shown that \( \exists \ l \ c \ l > 0 \) such that for sufficiently large \( y \) and infinitely many \( x \) we must have \( \pi(x + y) - \pi(x) - \pi(y) > cy/\ln^2(y) \).

By using sophisticated techniques H.L. Montgomery and R.C. Vaughan [6] proved that if \( M > 0 \) and \( N > 1 \) are integers then \( \pi(M + N) - \pi(M) \leq 2N/\ln(N) \). Now D.R. Heath-Brown and H. Iwaniec [7] show that if \( \theta > 11/20 \) and \( x \geq x(\theta) \) then \( \pi(x) - \pi(x - y) > y/(212 \ln(x)) \) in the range \( x^\theta \leq y \leq x/2 \). The methods used in this paper are elementary and give a different range of validity. The proofs of this paper use the following definitions and results.
\[ \pi(x) = \text{the number of primes } \leq x \]
\[ \text{Li}(x) = \int_2^x \frac{dt}{\ln(t)} \quad \text{for } x \geq 2 \]
\[ \text{Ls}(m) = \sum_{k=2}^{m} \frac{1}{\ln(k)} \quad \text{for any integer } m \geq 2 \]
\[ \pi(x) = \text{Li}(x) + O(xe^{-c\sqrt{x}}) \quad \text{for } x \geq 2, \ a > 0 \quad (1.3) \]
\[ \text{Li}(x) = x(1 + \sum_{k=1}^{n-1} \frac{k!}{\ln^k(x)})/\ln(x) + O(x/\ln^{n+1}(x)) \quad \text{for } x \geq 2 \quad (1.4) \]
\[ \pi(m) = \text{Ls}(m) + O(me^{-c\sqrt{m}}) \quad \text{for integer } m \geq 2, \ c > 0 \quad (1.5) \]
\[ 1 \text{ Li}(m) - \text{Ls}(m) < C \quad \text{for some constant } C \quad (1.6) \]

If the Riemann hypothesis holds, then (1.7) is true

\[ 1 \pi(x) - \text{Li}(x) < \sqrt{x} \ln(x) / 8\pi \quad \text{for } x \geq 2657 \quad (1.7) \]
\[ x(1 + 1/(2 \ln(x))) / \ln(x) < \pi(x) \quad \text{for } 59 \leq x \quad (1.8) \]
\[ \pi(x) < x(1 + 3/(2 \ln(x))) / \ln(x) \quad \text{for } 1 < x \quad (1.9) \]

Now (1.3), (1.4) can be found in Ayoub [8], whereas (1.5), (1.6) are found in T. Estermann [9]. Furthermore, the paper written by L. Schoenfeld [10] gives us (1.7). Finally (1.8), (1.9) were proven by J.B. Rosser and L. Schoenfeld [11].

2. THEOREMS, COROLLARIES AND THEIR PROOFS.

**THEOREM 1.** If 0 < d < 1 and x, y are sufficiently large with \( x \geq y \geq dx > 2 \), then

\[ \pi(x + y) - \pi(x) - \pi(y) = \pi(y)/\ln(x + y) < O(y/\ln^{n+1}(y)) \quad \text{for any natural number } n \geq 2. \]

**PROOF.** We have from (1.3) and (1.4) the following:

\[ \pi(x) = x/\ln(x) + x/\ln^2(x) + \cdots + (n-1)!x/\ln^n(x) + O(x/\ln^{n+1}(x)). \quad (2.1) \]

Now it is obvious that

\[ \pi(x+y) - \pi(x) = x/\ln(x+y) - x/\ln(x) + \sum_{k=1}^{n-1} \left( k!/\ln^k(x+y) - k!/\ln^{k+1}(x) \right) \]
\[ + y \left( 1 + \sum_{k=1}^{n-1} (k!/\ln^k(x+y)) \right) / \ln(x+y) + O \left( (x+y)/\ln^{n+1}(x+y) \right). \quad (2.2) \]

Given that \( x \geq 2, y > 0 \) then for \( 0 \leq k \leq n-1 \) we have

\( k!/\ln^{n+1}(x+y) < k!/\ln^{n+1}(x) \).

Hence (2.2) is replaced by

\[ \pi(x+y) - \pi(x) < y \left( 1 + \sum_{k=1}^{n-1} (k!/\ln^k(x+y)) \right) / \ln(x+y) + O \left( (x+y)/\ln^{n+1}(x+y) \right) \quad (2.3) \]

For \( k \geq 1 \), we observe that \( \ln^k(x+y) \geq \ln^k(2y) > \ln^k(y) \). Replacing \( \ln^k(x+y) \), (2.3) now becomes

\[ \pi(x+y) - \pi(x) < y \left( 1 + \sum_{k=1}^{n-1} (k!/\ln^k(y)) \right) / \ln(x+y) + O \left( (x+y)/\ln^{n+1}(y) \right). \quad (2.4) \]
Multiplying the first term on the right hand side of (2.4) by \(\ln(y)\ln(y)\) and using (2.1) we have replaced (2.4) by the following:

\[
\pi(x+y) - \pi(x) - \ln(y)\pi(y)\ln(x+y) < O\left(\frac{x+y}{\ln^m(y)}\right).
\]

It is obvious \(\exists\) a constant \(M > 0\) such that for \(x + y\) sufficiently large the left hand side of (2.5) is strictly less than

\[
M(x+y)/\ln^m(y).
\]

Since \(x \geq y \geq dx > 2\) for \(0 < d \leq 1\) then

\[
M(x+y)/\ln^m(y) < M(y/d + y)/\ln^m(y) < M'(y/\ln^m(y)).
\]

Hence by using (2.7) we conclude that

\[
\pi(x+y) - \pi(x) - \ln(y)\pi(y)/\ln(x+y) < O(y/\ln^m(y)).
\]

**THEOREM 2.** Let \(0 < q \leq 1\). If \(m, n\) are sufficiently large positive integers satisfying \(m \geq n \geq qm > 2\), then \(\pi(m+n) - \pi(m) < n/\ln(m+1) + Bne^{-\sqrt{\ln(2n)}}\) for \(B, a > 0\).

**PROOF.** By using (1.5) we see that

\[
\pi(m+n) - \pi(m) = \sum_{k=m+1}^{n} \left(1/\ln(k)\right) + O\left(\frac{n\ln(n)}{\ln(n)}\right).
\]

It is obvious that we can replace (2.8) by

\[
\pi(m+n) - \pi(m) - n/\ln(m+1) < O\left(\frac{n\ln(n)}{\ln(n)}\right).
\]

Now \(\exists\) a constant \(M > 0\) such that for \(m + n\) sufficiently large that the left hand side of (2.9) is strictly less than

\[
M(m+n)e^{-\sqrt{\ln(m+n)}}.
\]

Hence \(\pi(m+n) - \pi(m) < n/\ln(m+n) + Bne^{-\sqrt{\ln(2n)}}\).

**COROLLARY 1.** Let \(0 < q \leq 1\). If \(m, n\) are sufficiently large positive integers satisfying \(m \geq n \geq qm > 2\), then \(\pi(m+n) - \pi(m) < n/\ln(n)/\ln(m+1)\).

**PROOF.** By using the result of Theorem 2 with a slight modification we have

\[
\pi(m+n) - \pi(m) < n/\ln(n)/\ln(n)/\ln(m+1) + Bne^{-\sqrt{\ln(2n)}}.
\]

We rearrange the terms in (2.1) so that one can give an upper bound to replace \(n/\ln(n)\). With \(M > 0\), we now incorporate an upper bound of \(n/\ln(n)\) into (2.10) to establish that

\[
\pi(m+n) - \pi(m) < n/\ln(n)/\ln(n)/\ln(m+1) + Bne^{-\sqrt{\ln(2n)}}.
\]

Hence for \(n\) sufficiently large we have

\[
\pi(m+n) - \pi(m) < n/\ln(n)/\ln(m+1).
\]

**THEOREM 3.** Let \(0 < q \leq 1\). If \(m, n\) are sufficiently large positive integers satisfying \(m \geq n \geq qm > 2\), then \(\pi(m+n) - \pi(m) > n/\ln(m+n) - Ane^{-\sqrt{\ln(2n)}}\) for \(a > 0\) and \(A > 0\). constant \(M\) we have

\[
\pi(m+n) - \pi(m) > \sum_{k=m+1}^{n} \left(1/\ln(k)\right) - M(n+m)e^{-\sqrt{\ln(2n)}} - Mme^{-\sqrt{\ln(m)}}.
\]
With a slight modification in (2.11) and using another constant $M' > 0$ we see that
\[ \pi(m + n) - \pi(m) > n/\ln(n + m) - M'(m + n)e^{-\sqrt{\ln(n + m)}}. \] (2.12)

By rearranging the terms in (2.12) this will now become
\[ M'(m + n)e^{-\sqrt{\ln(n + m)}} > n/\ln(n + m) + \pi(m) - \pi(m + n). \] (2.13)

Since $m \geq n \geq qm > 2$ and $0 < q \leq 1$ then
\[ M'(m + n)e^{-\sqrt{\ln(n + m)}} < M'(n/q + n)e^{-\sqrt{\ln(n + m)}} = Ane^{-\sqrt{\ln(n + m)}}. \]

Hence $\pi(m + n) - \pi(m) > n/\ln(n + m) - Ane^{-\sqrt{\ln(n + m)}}$.

**COROLLARY 2.** Let $0 < q \leq 1$, $e > 0$. If $m, n$ are sufficiently large positive integers satisfying $m \geq n \geq qm > 2$, then $\pi(m + n) - \pi(m) > \ln(n)(\pi(n) - (1 + e)n/\ln^2(n)/\ln(n + m))$.

**PROOF.** By using the results of Theorem 3 with a slight modification we have
\[ \pi(m + n) - \pi(m) > n/\ln(n)(\ln(n + m))/\ln(n + m) - Ane^{-\sqrt{\ln(n + m)}}. \] (2.14)

Using an argument similar to that found in Corollary 1, we rearrange the terms in (2.1) so that one can give a lower bound to replace $n/\ln(n)$. With $D > 0$, we now incorporate a lower bound of $n/\ln(n)$ into (2.14) to establish the following
\[ \pi(m + n) - \pi(m) > \ln(n)/(\ln(n + m)/\ln(n + m)) - Ane^{-\sqrt{\ln(n + m)}}. \]

Hence for sufficiently large $n$
\[ \pi(m + n) - \pi(m) > \ln(n)(\ln(n) - (1 + e)n/\ln^2(n)/\ln(n + m)). \]

**THEOREM 4.** Let $1 \leq \theta < 2$. Let $m, n$ be positive integers with $m > 2657$ and $m \geq n \geq m^{1/9}$. If the Riemann hypothesis holds, then $\pi(m + n) - \pi(m) < n/\ln(n + m + 1) + \sqrt{n^8 + n} \ln(n^8 + n)/4\pi$.

**PROOF.** By using the upper and lower bounds of (1.7) we have
\[ \pi(m + n) - \pi(m) < Li(m + n) - Li(m) + (\sqrt{m + n} \ln(m + n) + \sqrt{m} \ln(m))/8\pi. \] (2.15)

Noting that $\sqrt{m + n} \ln(m + n) > \sqrt{m} \ln(m)$ and using (1.6), then (2.15) will now become
\[ \pi(m + n) - \pi(m) < \sum_{k=1}^{m+n} \left( 1/\ln(k) + \sqrt{m + n} \ln(m + n) / 4\pi. \right. \] (2.16)

It is obvious that we can replace (2.16) by
\[ \pi(m + n) - \pi(m) < n/\ln(n + m + 1) + \sqrt{m + n} \ln(m + n)/4\pi. \]

Given that $m \geq n \geq m^{1/9}$ for $1 \leq \theta < 2$ we may now conclude
\[ \pi(m + n) - \pi(m) < n/\ln(n + m + 1) + \sqrt{n^8 + n} \ln(n^8 + n)/4\pi. \]

**COROLLARY 3.** Let $1 \leq \theta < 2$. Let $m, n$ be positive integers with $m > 2657$, $n > 59$, and $m \geq n \geq m^{1/9}$. If the Riemann hypothesis holds, then $\pi(m + n) - \pi(m) < (\pi(n) - n(2\ln^2(n)))/\ln(n + m + 1) + \sqrt{n^8 + n} \ln(n^8 + n)/4\pi$.

**PROOF.** By using the result of Theorem 4 with a slight modification we have
\[ \pi(m + n) - \pi(m) < n/\ln(n)(\ln(n + m + 1)\ln(n)) + \sqrt{n^8 + n} \ln(n^8 + n)/4\pi. \] (2.17)

By rearranging (1.8) and incorporating it into (2.17) we achieve the following:
\[ \pi(m + n) - \pi(m) < \ln(n) \left( \pi(n) - n(2\ln^2(n)) \right) / (\ln(n + m + 1) + \sqrt{n^8 + n} \ln(n^8 + n))/4\pi. \]
THEOREM 5. Let $1 \leq \theta < 2$. Let $m, n$ be positive integers with $m > 2657$ and $m \geq n \geq m^{1/\theta}$. If the Riemann hypothesis holds then

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - n^{2\theta} + n \ln(n^\theta + n)/4\pi.$$

PROOF. By using the upper and lower bounds of (1.7) we have

$$\pi(m + n) - \pi(m) > \text{Li}(m + n) - \text{Li}(m) - (n^{\theta} + n \ln(m + n) + \sqrt{m + n} \ln(m))/8\pi. \tag{2.18}$$

Noting that $\sqrt{m + n} \ln(m + n) > \sqrt{m} \ln(m)$ and using (1.6), then (2.18) will now become

$$\pi(m + n) - \pi(m) > \sum_{k=m+1}^{n} (1/\ln(k)) - \sqrt{m + n} \ln(m + n)/4\pi. \tag{2.19}$$

It is obvious that we can replace (2.19) by

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - n^{2\theta} + n \ln(n^\theta + n)/4\pi.$$

COROLLARY 4. Let $1 \leq \theta < 2$. Let $m, n$ be positive integers with $m > 2657$ and $m \geq n \geq m^{1/\theta}$. If the Riemann hypothesis holds, then

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - \sqrt{m + n} \ln(n^\theta + n)/4\pi.$$

PROOF. By using the result of Theorem 5 with a slight modification we have

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - \sqrt{m + n} \ln(n^\theta + n)/4\pi. \tag{2.20}$$

By rearranging (1.9) and incorporating into (2.20) we achieve the following

$$\pi(m + n) - \pi(m) > n/\ln(n)(\pi(n) - 3n/(2 \ln^2(n)))/\ln(m + n) - \sqrt{n^{\theta} + n} \ln(n^\theta + n)/4\pi.$$

3. FINAL COMMENTS.

I feel that Theorem 1 and the Corollaries 1 and 3 are relevant to the disagreement between Erdős and Richards in their paper [4] dealing about whether the following conjecture is true.

$$\pi(x + y) - \pi(x) - \pi(y) < cy / \ln^2(y). \tag{3.1}$$

Of course, Theorem 1 states that (3.1) is true provided that for $0 < d \leq 1$, $x$ and $y$ are sufficiently large and $x \geq y \geq dx > 2$. Under similar restrictions, Corollary 1 also states that (3.1) is true. Moreover, if we assume the conditions that are given in the Corollary 3 then we can give explicit bounds for which (3.1) is correct.

As for the mysterious person who told P. Erdős [12] that the "correct" conjecture should be $\pi(x + y) \leq \pi(x) + 2\pi(y/2)$, I claim to have made some progress in this direction. From Rosser, Schoenfeld and Yohe [13] we have $\pi(2x) - \pi(x) < \pi(x)$. If $m \geq n$ then $\ln(n) \pi(n)/\ln(m + 1) < \pi(n) < 2\pi(n/2)$. Hence with the restrictions found in the Corollary 1 we have $\pi(m + n) \leq \pi(m) + 2\pi(n/2)$.

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