COMPLETENESS OF REGULAR INDUCTIVE LIMITS

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ABSTRACT. Regular LB-space is fast complete but may not be quasi-complete. Regular inductive limit of a sequence of fast complete, resp. weakly quasi-complete, resp. reflexive Banach spaces is fast complete, resp. weakly quasi-complete, resp. reflexive complete, space.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, complete, quasi-complete, fast complete space.

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1. INTRODUCTION.

In [1, §31.6] K"{o}the has a sequence of Banach spaces $E_1 \subset E_2 \subset \ldots$ whose inductive limit is not quasi-complete. In [2] there is an example of reflexive Frechet spaces $E_n$ whose inductive limit is not even fast complete. Since an LF-space is fast complete iff it is regular, see [3], there is a natural question asked by Jorge Mujica in [4]: Is every regular LB-space complete?

Throughout the paper $E_1 \subset E_2 \subset \ldots$ is a sequence of locally convex spaces with continuous inclusions $E_n \rightarrow E_{n+1}$, $n \in \mathbb{N}$. Their locally convex inductive limit is denoted by $E$. The space $E$ is called regular if every set bounded in $E$ is bounded in some $E_n$.

2. MAIN RESULTS.

Let $F$ be a locally convex space and $A \subset F$ absolutely convex. We denote by $F_A$ the seminormed space $U\{nA; n \in \mathbb{N}\}$ whose topology is generated by the Minkowski functional of $A$. If $F_A$ is Banach space, $A$ is called Banach disk. The space $F$ is called fast complete if every set bounded in $F$ is contained in a bounded Banach disk. Every sequentially complete space is fast complete and there are fast complete spaces which are sequentially incomplete, see [5].

EXAMPLE. For each $n \in \mathbb{N}$ and $x : \mathbb{N} \times \mathbb{N} \rightarrow C$, put

$$
\|x\|_n = \max \{\sup_{i \leq n} |x_{ij}|; i \leq n, j \in \mathbb{N}\}, \sup \{|x_{ij}|; i > n, j \in \mathbb{N}\},
$$

$E_n = \{x; \|x\|_n < \infty \land \lim_{j \rightarrow \infty} x_{ij} = 0 \text{ for } i > n\}$,

$B_n = \{x \in E_n; \|x\|_n \leq 1\}$, and
We prove that each \( E_n \) is a Banach space, \( E_1 \subset E_2 \subset \ldots \), inclusions \( E_n \to E_{n+1}, \ n \in \mathbb{N} \), are continuous, \( E \) is regular and not quasi-complete.

CLAIM 1. Each space \( E \) is Banach.

PROOF. Let \( \{x(k)\} \) be a Cauchy sequence in \( E \). For each \( i, j \in \mathbb{N} \) the sequence \( \{x(k)_{ij}\} \) is Cauchy in \( C \) and has a limit \( x_{ij} \). Let \( x \) be the matrix with the entries \( x_{ij} \).

Given \( \varepsilon > 0 \), there is \( k \) such that \( p, r \geq k \) implies \( \|x(p) - x(r)\|_n \leq \varepsilon \). Hence,

\[
\|x(p) - x\|_n \leq \limsup_{r \to \infty} \|x(p) - x(r)\|_n \leq \varepsilon \text{ and } \|x(p) - x\|_n \leq \varepsilon.
\]

Take \( i > n \) and choose \( j_i \) so that \( |x(k)_{ij}| < \varepsilon \) for \( j > j_i \).

Then

\[
|x_{ij}| \leq |x_{ij} - x(k)_{ij}| + |x(k)_{ij}| \leq \|x - x(k)\|_n + |x(k)_{ij}| < 2\varepsilon \text{ and } \lim_{j \to \infty} x_{ij} = 0.
\]

CLAIM 2. \( E_1 \subset E_2 \subset \ldots \) and each inclusion \( E_n \to E_{n+1} \) is continuous. Proof follows from the inequalities \( \|x\|_1 \geq \|x\|_2 \geq \ldots, x \in \bigcup \{E_n; n \in \mathbb{N}\} \).

CLAIM 3. \( E \) is regular.

PROOF. Let \( D \subset E \) be not bounded in any \( E_n \). For each \( n \in \mathbb{N} \) choose \( x(n) \in D \) such that \( \|x(n)\|_n > n \). There are \( i(n), j(n) \in \mathbb{N} \) for which

\[
|x(n){i(n),j(n)}| > n j(n) i(n) \text{ if } i(n) \leq n \\
|x(n){i(n),j(n)}| > n i(n) j(n) \text{ if } i(n) > n
\]

Put \( m(n) = n + \max\{i(k); k \leq n\} \) and \( r(n) = \min\{j(k)-i(k); k \leq m(n)\}, n \in \mathbb{N} \).

If \( k > n \) then \( \|x(n)\|_{m(k)} \geq |x(n){i(n),j(n)}| > r\varepsilon(n) \).

if \( k \leq n \) then \( \|x(n)\|_{m(k)} \geq \|x(n)\|_{m(n)} > r\varepsilon(n) \).

Let \( V = \bigcup (r(k)B_{m(k)}; k \in \mathbb{N}) \) and \( U = \overline{V} \). Assume \( x(n) \in n U \). Then

\[
x(n) = \sum_{k=1}^{\infty} a_k y(k),
\]

where \( a_k \geq 0, \sum a_k = 1, \) and \( y(k) \in r\varepsilon(n) B_{m(k)} \). To prove that \( |y(k){i(n),j(n)}| \leq n \)

for \( k \in \mathbb{N} \), we have to distinguish three cases:

(a) \( k > n \): Then \( |y(k){i(n),j(n)}| = |y(k){i(n),j(n)}| j(n)^{i(n)-i(n)} \leq \)

\[
\leq \|y(k)\|_{m(k)} j(n)^{i(n)} \leq r\varepsilon(n) j(n)^{i(n)} \leq n.
\]
(b) \( k \leq n \& i(n) \leq m(k) \): Then \(|y(k)_{i(n),j(n)}| \leq l y(k)_{m(k),j(n)}i(n) \leq nr(k)j(n)i(n) \leq n.\)

(c) \( k \leq n \& i(n) > m(k) \): Then \(|y(k)_{i(n),j(n)}| \leq l y(k)_{m(k),j(n)}i(n) \leq nr(k)j(n)i(n) \leq n.\)

On the other hand \(|x(n)_{i(n),j(n)}| > n\) and \(x(n)\) cannot be a convex combination of \(y(k), k \leq s, \) i.e. \(x(n) \notin nU.\) Since \(U\) is a 0-neighborhood in \(E, D\) is not bounded in \(E.\)

CLAIM 4. \(E\) is not quasi-complete.

PROOF. Let \(\Delta = \{\delta \subset \N \times \N; \{j \in \N; (i,j) \in \delta\} \text{ is finite, } i \in \N\} \) be ordered by set inclusion. Denote by \(x(\delta)\) the set characteristic function of \(\delta \in \Delta.\) Then \(x(\delta); \delta \in \Delta\) \(\subset B_1\) and the filter associated with \(\delta \to x(\delta)\) is bounded in \(E_1,\) hence also bounded in \(E.\)

Let \(P_n: C^{\N \times \N} \to C^{\N \times \N}\) be the projection of an \(N \times N\) matrix on its \(n\)-th row. Take a closed absolutely convex 0-neighborhood \(V\) in \(E.\) For each \(n \in \N\) choose \(m(n) \in \N\) and \(r(n) > 0\) so that \(r(n)B_n \subset V, m(n) \geq 2r(n)^{-1}/n,\) and put \(\sigma = \{(i,j) \in \N \times \N; j \leq m(i)\}.\)

If \(\gamma, \delta \in \Delta, \gamma, \delta \geq \sigma,\) then \(x(\gamma)_{ij} - x(\delta)_{ij} = 0\) for \(j \leq m(i)\) and \(IP_n(x(\gamma) - x(\delta)) = \sup (j^{-n}| x(\gamma)_{nj} - x(\delta)_{nj} |; j > m(n)) < m(n)^{-n} \leq 2^{-n} r(n).\) Hence \(2^n IP_n(x(\gamma) - x(\delta)) \in r(n)B_n \subset V.\) Since \(V\) is absolutely convex, the sequence

\[ y_k = \sum_{n=1}^{k} 2^{-n} 2^{n} IP_n(x(\gamma) - x(\delta)), k \in \N \]

is contained in \(V.\) It is also contained in \(B_1\) and converges coordinate-wise to \(x(\gamma) - x(\delta)\) in \(E_1.\) Hence \(x(\gamma) - x(\delta)\) is in the weak closure of \(V.\) Since \(V\) is closed and convex, it is also weakly closed and \(x(\gamma) - x(\delta) \in V.\) So \(\{x(\delta); \delta \in \Delta\}\) is a base of a bounded Cauchy filter in \(E.\) If it had a limit \(x \in E,\) then \(x_{ij} = 1\) for all \(i, j \in \N.\) This would imply \(x \notin \N\) for any \(n \in \N\) and \(x \notin E, \) q.e.d.

LEMMA. Regular inductive limit of a sequence of semireflexive, resp. reflexive, spaces is semireflexive, resp. reflexive.

PROOF. Let each \(E_n\) be semireflexive. Since \(E = indlim E_n\) is regular, its strong dual \(E_n^d\) equals to \(projlim \langle E_n \rangle^d\) and \((E_n^d)^d \subset U \{\langle (E_n^d)^d \rangle^d; n \in \N\} = \U \{E_n^d; n \in \N\} = E.\)

Let each \(E_n\) be reflexive. By [7;IV, 5.6] it suffices to show that \(E\) is semireflexive and barreled. Take a barrel \(B\) in \(E.\) For each \(n \in \N, B \cap E_n\) is a barrel in \(E_n.\) Since \(E_n\) is reflexive, the barrel \(B \cap E_n\) is a neighborhood in \(E_n,\) which implies that \(B\) is a neighborhood in \(E\) and \(E\) is barreled.

CONSEQUENCE. Inductive limit of a sequence of reflexive Banach spaces is reflexive.

PROOF. By [6; Th. 4] the inductive limit of reflexive Banach spaces is regular.
THEOREM. Let $E = \text{indlim} E_n$ be regular. Then:

(a) Each $E_n$ fast complete $\Rightarrow E$ fast complete.

(b) Each $E_n$ weakly quasi-complete $\Rightarrow E$ weakly quasi-complete.

(c) Each $E_n$ semireflexive $\Rightarrow E$ quasi-complete.

(d) Each $E_n$ reflexive Banach $\Rightarrow E$ complete.

PROOF.

(a) Let $B \subset E$ be bounded, then it is bounded in some $E_n$ and contained in a bounded Banach disk in $E_n$. Since any Banach disk bounded in $E_n$ is also bounded in $E$, the proof is complete.

(b) Follows from Lemma since any locally convex space is weakly quasi-complete iff it is semireflexive, [7;IV, 5.5].

(c) Follows from (b) since every weakly quasi-complete space is quasi-complete.

(d) Let $\mathcal{F}$ be a Cauchy filter in $E$. Then $\mathcal{F}$ as a filter of continuous linear functionals on $E'_b$, converges uniformly on bounded sets in $E'_b$ to a linear, not necessarily continuous, functional $h : E'_b \to C$. Since $E$ is reflexive, it suffices to show that $h$ is continuous.

Let $B = \{x_n : n = 0, 1, 2, \ldots\} \subset h^{-1}(0)$ be a bounded set in $E'_b$. We have to show that $h(x_0) = 0$. Choose $\varepsilon > 0$. The set $B = \{x_n : n = 0, 1, 2, \ldots\}$ is bounded in $E'_b$, hence there is $F \in \mathcal{F}$ such that

$$\sup \{|f(x_n) - h(x_n)| ; f \in F, x_n \in B\} < \varepsilon.$$ Fix an $f \in F$ and choose $n \in \mathbb{N}$ so that $|f(x_n) - f(x_0)| < \varepsilon$. Then $|h(x_0)| = |h(x_0) - h(x_n)| + |h(x_n) - f(x_0)| + |f(x_0) - f(x_n)| < 3\varepsilon$, which implies $h(x_0) = 0$.

CONJECTURE. Regular LB-space may not be sequentially complete.

REFERENCES