BOUNDED SETS IN $\mathcal{L}(E,F)$

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Abstract: Let $E$ and $F$ be Hausdorff locally convex spaces, and let $\mathcal{L}(E,F)$ denote the space of continuous linear maps from $E$ to $F$. Suppose that for every subspace $N \subset E$ and an absolutely convex set $A \subset E$ which is bounded, closed, and absorbing in $N$, there is a barrel $D \subset E$ such that $A \subset D \cap N$. Then it is shown that the families of weakly and strongly bounded subsets of $\mathcal{L}(E,F)$ are identical if and only if $E$ is locally barreled.

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I. INTRODUCTION.

Throughout this paper $E$ and $F$ will denote Hausdorff locally convex spaces, and $\mathcal{L}(E,F)$ the space of continuous linear maps from $E$ to $F$. An absolutely convex set $A$ in $E$ will be called a disk. If $A$ is any subset of $E$, its linear hull will be denoted by $E_A$. For a disk $B$ in $E$, its linear hull is given by $E_B = \bigcup \{nB : n \geq 1\}$. Equipped with the topology generated by the Minkowski functional of $B$, $E_B$ is a semi-normed space. This leads to the definition which follows.

DEFINITION 1: Let $B \subset E$ be a disk. If $E_B$ is a barreled normed space, then $B$ is called a barreled disk; $E$ is locally barreled if each bounded set in $E$ is contained in a closed, bounded barreled disk.
II. A UNIFORM BOUNDEDNESS THEOREM

It is proven in [1] that in a locally convex space $E$ the families of $\sigma(E', E)$-bounded and $\beta(E', E)$-bounded sets are the same if $E$ is locally barreled. This is proven for the general case $L(E, F)$ in our first result below. Conversely, in section III. we will examine the local barreledness of $E$ in terms of subsets of $L(E, F)$ which are bounded for any $S$-topology, where $S$ is a family of bounded sets which covers $E$.

**THEOREM 2:** If $E$ is locally barreled then the families of bounded sets in $\mathcal{L}(E, F)$ are the same for all $S$-topologies, where $S$ is a family of bounded sets in $E$ which covers $E$.

**PROOF:** Assume $E$ to be locally barreled. Let $V$ be a closed, absolutely convex $0$-neighborhood in $F$. Let $H \subset \mathcal{L}(E, F)$ be pointwise bounded. Let

$$D = \bigcap\{u^{-1}(V) : u \in H\}.$$  

Then $D$ is a closed disk in $E$. Since $H$ is pointwise bounded, we have:

$$x \in E \Rightarrow \bigcup\{u(x) : u \in H\} \subset \alpha V,$$

for some $\alpha > 0$. By taking inverse images, it follows that $D$ is absorbing in $E$; hence, $D$ is a barrel in $E$. In 8.5, Chapter II of [2] it is proven that $D$ absorbs all bounded Banach disks. A careful reading of that proof reveals that the only property of Banach spaces which is used is the property of being barreled. Hence, any barrel in $E$ absorbs all closed, bounded barreled disks in $E$, as well. Moreover, if $A$ is any bounded subset of $E$, then $A$ is contained in some closed, bounded barreled disk $B$. Therefore, $D$ absorbs $A$ and 3.3, Chapter III of [2] now asserts that $H$ is bounded for the topology of bounded convergence on $\mathcal{L}(E, F)$. \qed

III. LOCALLY BARRELED SPACES AND BOUNDED SETS IN $\mathcal{L}(E, F)$.

Let $(P)$ denote the following property of a locally convex space $E$:

$(P)$ For each absolutely convex, closed, bounded set $A \subset E$ there exists a barrel $D \subset E$ such that $A = D \cap E_A$.

**THEOREM 3:** Let $E$ and $F$ be a Hausdorff locally convex spaces. Assume $E$ satisfies property $(P)$. Then the following are equivalent:
(a) The families of bounded subsets of \( L(E, F) \) are identical for all \( S \)-topologies on \( L(E, F) \), where \( S \) is a family of bounded subsets of \( E \) which covers \( E \).

(b) \( E \) is locally barrelled.

**Proof.** In view of Theorem 2, we need only prove (a) \( \Rightarrow \) (b).

If \( E \) is not locally barrelled, then there exists an absolutely convex, closed, bounded set \( B \subset E \) such that \( E_B \) is not barrelled. We will first show that every set \( M \) which is closed and bounded in \( E_B \) is also closed in \( E \). Denote by \( M_0 \) the closure of \( M \) in \( E \). Since \( M \) is bounded in \( E_B \), \( M \subset \lambda B \), for some \( \lambda > 0 \). \( \lambda B \) is closed in \( E \). Hence \( M_0 \subset \lambda B \subset E_B \). Take \( x_0 \in M_0 \) and a net \( \eta \subset M \) such that \( \eta \rightarrow x_0 \) in the topology of \( E \). The identity \( id : E_B \rightarrow E \) is continuous, and \( \{k^{-1}B : k \in \mathbb{N}\} \) is a basis for the neighborhoods of zero in \( E_B \) consisting of sets closed in \( E \). Therefore, by 3.2.4 of [3], \( \eta \rightarrow x_0 \) in the topology of \( E_B \). Finally, \( M \) is closed in \( E_B \). Hence \( x_0 \in M \), so \( M \) is closed in \( E \).

Now choose a barrel \( A \) in \( E_B \) which is not a 0-neighborhood in \( E_B \). Then we may choose a sequence \( \{x_n\} \subset E_B \setminus A \) such that \( x_n \rightarrow 0 \) in the topology of \( E_B \). The normability of \( E_B \) implies that \( \{x_n\} \) is locally convergent; thus we may choose a sequence \( \{a_n\} \) of positive real numbers such that \( a_n \uparrow \infty \) and \( a_n x_n \rightarrow 0 \) in the normed space \( E_B \). Since the normed topology of \( E_B \) is finer than the topology on \( E_B \) induced by \( E \), the sequence \( \{a_n x_n\} \) also converges to 0 with respect to the topology of \( E \). This means

\[
S = \{a_n x_n : n \in \mathbb{N}\}
\]

is bounded in \( E \).

Since \( A \cap B \) is absolutely convex, bounded, and closed in \( E_B \), it is also closed and bounded in \( E \). By (P), there is a barrel \( D \subset E \) such that

\[
A \cap B = D \cap E_A = D \cap E_B.
\]

Now, \( x_n \notin D \) for each \( n \), and we may therefore choose \( f_n \in E' \) such that \( |f_n(x)| \leq 1 \) for any \( x \in D \) while \( f_n(x_n) = 1 \), where each \( f_n \) is real valued.

Let \( y_0 \in F \setminus \{0\} \), and define \( g : \mathbb{R} \rightarrow F \) by

\[
g(z) = z y_0,
\]

for each \( z \in \mathbb{R} \). \( g \) is a linear map taking bounded sets in \( \mathbb{R} \) to bounded sets in \( F \); therefore, \( g \) is continuous.
Now, for each \( n \in \mathbb{N} \), define \( h_n : E \rightarrow F \) by
\[ h_n = g \circ f_n. \]
As the composition of two linear, continuous maps, each \( h_n \in \mathcal{L}(E, F) \).

Put
\[ H = \{ h_n : n \in \mathbb{N} \}. \]

First, notice that for each \( x \in D \), \( |f_n(x)| \leq 1 \), hence \( h_n(x) \in C \), where \( C \) is the line segment from \(-y_0\) to \( y_0 \) in \( F \). Obviously, \( C \) is bounded in \( F \); consequently,
\[ \bigcup \{ h_n(x) : n \in \mathbb{N} \} \]
is bounded in \( F \) for each \( x \in D \). Since \( D \) is absorbing in \( E \),
\[ \bigcup \{ h_n(x) : n \in \mathbb{N} \} \]
is bounded in \( F \) for each \( x \in E \) as well; this makes \( H \) a pointwise bounded set.

Finally,
\[ \bigcup \{ h_n(x) : x \in S, n \in \mathbb{N} \} = \bigcup \{ h_n(a_n x_n) : n \in \mathbb{N} \} = \bigcup \{ a_n g(1) : n \in \mathbb{N} \} = \bigcup \{ a_n \{ y_0 \} : n \in \mathbb{N} \}. \]
Letting \( \alpha_n = a_n^{-1} \), then
\[ \lim_{n \to \infty} \alpha_n = 0, \]
while
\[ \lim_{n \to \infty} \alpha_n(a_n \{ y_0 \}) = y_0 \neq 0. \]
This means \( H(S) \) is not bounded in \( F \); thus \( H \) is not bounded for the topology of uniform convergence on bounded sets. \( \square \)

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References

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