EIGENFUNCTION EXPANSION FOR A REGULAR FOURTH ORDER EIGENVALUE PROBLEM WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITIONS

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1. INTRODUCTION.

The regular right-definite eigenvalue problems for second order differential equations with eigenvalue parameter in the boundary conditions, have been studied in Walter [1], Fulton [2] and Hinton [3].

The object of this paper is to prove the expansion theorem for the following regular fourth order eigenvalue problem:

\[
\begin{align*}
  \nu: & \quad (Ku'')'' + (Pu')' + qu = \lambda u, \quad x \in [a, b] \\
  u(a) &= (Pu')(a) = (Ku'')(a) = 0 \\
  (Ku''')(b) &= (Pu')(b) = -\lambda u(b)
\end{align*}
\]  

(1.1)

where \( P, q \) and \( K \) are continuous real-valued functions on \([a, b]\). We assume that \( P(x) > 0, q(x) > 0, \) and \( K(x) > 0 \) while \( \lambda \) is a complex number.

Recently, Zayed [4] has studied the special case of the problem (1.1) wherein \( K(x) = \alpha^2 \), \( \alpha^2 \) is a constant and \( q(x) = 0 \).

Further, problem (1.1), in general, describes the transverse motion of a rotating beam with tip mass, such as a helicopter blade (Ahn [5]) or a bob pendulum suspended from a wire (Ahn [6]).
Ahn [7] has shown that the set of eigenvalues of problem (1.1) is not empty, has no finite accumulation points and is bounded from below. He used an integral-equation approach.

In this paper, our approach is to give a Hilbert space formulation to the problem (1.1) and self-adjoint operator defined in it such that (1.1) can be considered as the eigenvalue problem of this operator.

2. HILBERT SPACE FORMULATION.

We define a Hilbert space $H$ of two-component vectors by

$$H = L^2(a,b) \oplus \mathbb{C};$$

with inner product

$$\langle f, g \rangle = \int_a^b f_1 \overline{g_1} dx + f_2 \overline{g_2}, f, g \in H$$

and norm

$$\|f\|_H^2 = \int_a^b |f_1|^2 dx + |f_2|^2$$

where

$$f = (f_1, f_2) = (f_1(x), f_1(b)) \in H$$

and

$$g = (g_1, g_2) = (g_1(x), g_1(b)) \in H.$$

We can define a linear operator $A: D(A) \rightarrow H$ by

$$Af = (\tau f_1, - (Kf_1'')(b) + (Pf_1')(b)) \quad \forall f = (f_1, f_2) \in D(A)$$

where the domain $D(A)$ of $A$ is a set of all $f = (f_1, f_2) \in H$ which satisfy the following:

(i) $f_1, f_1', f_1''$ and $f_1'''$ are absolutely continuous with

$$\tau f_1 \in L^2(a,b) \text{ and } \int_a^b (K|f_1''|^2 + P|f_1'|^2 + q|f_1|^2) dx < \infty.$$

(ii) $f_1(a) = (Pf_1')(a) = (Kf_1'')(a) = 0$

(iii) $f_2 = f_1(b)$.

REMARK 2.1. The parameter $\lambda$ is an eigenvalue of (1.1) and $f_1$ is a corresponding eigenfunction of (1.1) if and only if

$$f = (f_1, f_1(b)) \in D(A) \quad \text{and} \quad Af = \lambda f$$

Therefore, the eigenvalues and the eigenfunctions of problem (1.1) are equivalent to the eigenvalues and the eigenfunctions of operator $A$. 


We consider the following assumptions:

(1) \[ \lim_{x \to b} [K'(x)f_1(x) - K(x)f_1'(x)] = 0, \]  

(2.5) \[ \lim_{x \to b} [K'(x)g_1(x) - K(x)g_1'(x)] = 0. \]

**Lemma 2.1.** The linear operator \( A \) in \( H \) is symmetric.

**Proof.** On using the boundary conditions of (1.1) we get,

\[
\langle Af, g \rangle = \int_a^b (tf_g)\overline{g_1} \, dx + \int_a^b (Kf_1')(b) + (Pf_1')(b)\overline{g_1}(b)
\]

\[
= \int_a^b (Kf_1''(b) + (Pf_1')(b)\overline{g_1}(b) + \int_a^b \overline{g_1}(b) dx - (Kf_1''(b))(b)\overline{g_1}(b)
\]

\[
+ (Pf_1')(b)\overline{g_1}(b)
\]

Integrating the first term of (2.6) by parts four times and integrating the second term of (2.6) by parts twice, we get

\[
\langle Af, g \rangle = \int_a^b f_1[(\overline{Kg_1''}) - (\overline{Pg_1'}) + \overline{g_1}] dx + f_1(b)\overline{g_1}(b)\]

\[
+ f_1''(b) [K'(b)\overline{g_1}(b) - K(b)\overline{g_1}(b)] - \overline{g_1}(b) [K'(b)f_1(b) - K(b)f_1'(b)]
\]

Applying the conditions (2.5) and using the boundary conditions of (1.1), we obtain

\[
\langle Af, g \rangle = \int_a^b f_1(\overline{Kg_1''}) dx + f_1(b)\overline{g_1}(b)\]

\[= \langle f, Ag \rangle. \]

**Remark 2.2.** For all \( f = (f_1, f_2) \) in \( D(A) \) and \( f_2 = f_1(b) \neq 0 \), the domain \( D(A) \) is dense in \( H \).

Since the operator \( A \) in \( H \) is symmetric and dense in \( H \), \( A \) is self-adjoint.

### 3. The Boundedness.

We shall show that the self-adjoint operator \( A \) is unbounded from above and bounded from below. We also show that \( A \) is strictly positive.

**Lemma 3.1.**

(i) If \( f, f' \) are absolutely continuous with \( f(a) = 0 \) and \( P(x) > 0 \) in \( [a, b] \), then we have \( P(x) > c_1 \) for some constant \( c_1 > 0 \) such that

\[
\frac{b}{a} \int P(x)|f'(x)|^2 \, dx > c_1 |f(b)|^2.
\]

(ii) For \( f \in C^2[a, b] \), there exists a positive constant \( c_2 \) such that

\[
\frac{b}{a} \int |f(x)|^2 \, dx < c_2 \frac{b}{a} \int |f''(x)|^2 \, dx
\]
(i) Since $P(x) > 0$ in $[a,b]$, we have $P(x) > c_1$ for some $c_1 > 0$.

Consequently, on using Schwartz's inequality, we get

$$
\int_a^b P(x)|f'(x)|^2 \, dx > c_1 \int_a^b |f'(x)|^2 \, dx > c_1 \int_a^b |f'(x)|^2 \, dx > c_1 |f(b)|^2
$$

where $\int_a^b f'(x)dx = f(b) - f(a) = f(b)$, since $f(a) = 0$.

(ii) By using Theorem 2 in [8, p.67], we have for $f(x) \in C^1[a,b]$,

$$
\int_a^b |f(x)|^2 \, dx < 4(b-a)^2 \int_a^b \left| \frac{df(x)}{dx} \right|^2 \, dx
$$

Since $\int_a^b \left| \frac{df(x)}{dx} \right|^2 \, dx < 4 \int_a^b \left| \frac{df(x)}{dx} \right|^2 \, dx$, we get

$$
\int_a^b |f(x)|^2 \, dx < 4(b-a)^2 \int_a^b |f''(x)|^2 \, dx
$$

(3.1)

Applying (3.1) again for $|f'(x)|$, we get

$$
\int_a^b |f'(x)|^2 \, dx < 16(b-a)^2 \int_a^b |f''(x)|^2 \, dx
$$

(3.2)

from (3.1) and (3.2) we get

$$
\int_a^b |f(x)|^2 \, dx < c_2 \int_a^b |f''(x)|^2 \, dx \text{ where the constant } c_2 = 256(b-a)^4.
$$

LEMMA 3.2. The linear operator $A$ is bounded from below.

**PROOF.** On using the boundary conditions of (1.1) we get

$$
\langle Af, f \rangle = \int_a^b (\tau f_1) \overline{f_1} \, dx + \left[ -(Kf''_1)(b) + (Pf'_1)(b) \right] \overline{f_1(b)}
$$

$$
\int_a^b (Kf''_1) \overline{f_1} \, dx - \int_a^b (Pf'_1) \overline{f_1} \, dx + \int_a^b qf_1 \overline{f_1} \, dx - (Kf''_1)(b) \overline{f_1(b)} + (Pf'_1)(b) \overline{f_1(b)}.
$$

(3.3)

Integrating (3.3) by parts twice and using the boundary conditions of (1.1), we obtain

$$
\langle Af, f \rangle = \int_a^b (Kf''_1)(b) \overline{f_1(b)} - K(b) \overline{f_1'(b)} + \int_a^b K|f''_1|^2 \, dx
$$

$$
+ \int_a^b |f_1''|^2 \, dx + \int_a^b q|f_1|^2 \, dx.
$$

On using (2.5) (ii) and lemma (3.1), we get

$$
\langle Af, f \rangle > \int_a^b \frac{K(x)}{c_2} |f_1(x)|^2 \, dx + c_1 |f_1(b)|^2 + \int_a^b q(x) |f_1(x)|^2 \, dx
$$

$$
\int_a^b \frac{K(x)}{c_2} + q(x) |f_1(x)|^2 \, dx + c_1 |f_2|^2
$$
where
\[ c_3 = \inf_{x \in [a,b]} \left( \frac{K(x)}{c_2} + q(x) \right) \]
Therefore
\[ \langle Af, f \rangle > c \| f \|^2 \quad (3.4) \]
where the constant \( c = \min (c_3, c_1) \).

It follows, from (3.4), that the operator \( A \) is bounded from below.
Since \( c_1 > 0, K(x) > 0, q(x) > 0, c_2 > 0 \) and \( c = \min (c_3, c_1) \) then the constant \( c \) is positive (\( c > 0 \)) and hence \( A \) is strictly positive.

REMARK 3.1.
(i) Since \( A \) is a symmetric operator (from lemma 2.1) then \( A \) has only real eigenvalues.
(ii) By Lemma 3.2, we deduce that the set of all eigenvalues of \( A \) is also bounded from below.
(iii) Since \( A \) is strictly positive, then the zero is not an eigenvalue of \( A \).

By using theorem 3 in [8, p.60] we can state that:
Since \( A \) in \( H \) is symmetric and bounded from below, then for every eigenvalue \( \lambda \) of \( A \) in \( H \), \( \lambda > c \) where the constant \( c \) is the same as in (3.4). This means that \( 0 < c < \lambda_1 < \lambda_2 < \ldots \ldots < \lambda_1 \) according to the size and \( \lambda_1 \rightarrow +\infty \) as \( i \rightarrow \infty \).
This implies that the set of all eigenvalues of \( A \) is unbounded from above.

REMARK 3.2. Since the operator \( A \) is self-adjoint, then \( A \) has only real eigenvalues and the eigenfunctions of \( A \) are orthonormal. By using theorem 3 in [8, p.30], the density of the domain \( D(A) \) in \( H \) gives us the completeness of the orthonormal system of eigenfunctions \( \phi_1, \phi_2, \phi_3, \ldots \) of \( A \).

4. THE EIGENFUNCTIONS OF THE OPERATOR \( A \).

We suppose \( \phi_\lambda(x), \psi_\lambda(x), \chi_\lambda(x) \) and \( \gamma_\lambda(x) \), where \( \lambda \in \mathbb{C} \) is not an eigenvalue of \( A \), are the fundamental set of solutions of the fourth order differential equation of (1.1) with the initial conditions:

\[ \phi_\lambda(a) = 0, \quad (P\phi_\lambda)(a) = 0, \quad \phi_\lambda'(a) = 1, \quad (K\phi_\lambda'')(a) = 0 \quad (4.1) \]
\[ \psi_\lambda(a) = 0, \quad (P\psi_\lambda)(a) = 0, \quad \psi_\lambda'(a) = 0, \quad (K\psi_\lambda'')(a) = 1 \quad (4.2) \]
\[ \chi_\lambda(b) = 0, \quad (P\chi_\lambda')(b) = 1, \quad \chi_\lambda''(b) = 0, \quad (K\chi_\lambda'')(b) = 1 \quad (4.3) \]
\[ \gamma_\lambda(b) = 1, \quad (P\gamma_\lambda')(b) = 1 + \lambda, \quad \gamma_\lambda''(b) = 0, \quad (K\gamma_\lambda'')(b) = 1 \quad (4.4) \]

Therefore the Wronskian is
\[ W = \lim_{x \rightarrow b} \left( x_\lambda(x)(Py_\lambda')(x) - (Px_\lambda')(x) \gamma_\lambda(x) \right) = -1 \neq 0 \]
Thus the solutions $\phi_1(x), \psi_1(x), \chi_1(x)$ and $\gamma_1(x)$ are linearly independent of $\tau u = \lambda u$. Putting $x = b$, we obtain the Wronskian in the form:

$$ W = \psi''(b) [\lambda \phi_1(b) - (P\phi_1)'(b) + (K\psi''_1)(b)] $$

$$ - \psi'(b) [\lambda \psi_1(b) - (P\psi_1)'(b) + (K\psi''_1)(b)] 
eq 0 \quad (4.5) $$

Now, for $f = (f_1, f_2) \in H$, we define $\phi = (\phi_1, \phi_2) \in D(A)$ as the unique solution of $(\lambda I - A)\phi = f$.

Application of variation of parameter method yields the unique solution $\phi \in D(A)$ of $(\lambda I - A)\phi = f$, $f \in H$ with:

$$(\lambda I - \tau) \phi_1 = f_1$$

$$\lambda \phi_1(b) - (P\phi_1)'(b) + (K\psi''_1)(b) = f_2$$

Therefore

$$ \phi_1(x) = \int_a^b \left[ \frac{\chi_1(x)\alpha_3(t) + \gamma_1(x)\alpha_4(t)}{W} \right] f_1(t)dt $$

where

$$ \alpha_1(t) = \left[ \begin{array}{ccc} \psi_1(t) & \chi_1(t) & \gamma_1(t) \\ \psi_1'(t) & \chi_1'(t) & \gamma_1'(t) \\ \psi_1''(t) & \chi_1''(t) & \gamma_1''(t) \end{array} \right] $$

$$ \alpha_2(t) = \left[ \begin{array}{ccc} \phi_1(t) & \psi_1(t) & \gamma_1(t) \\ \phi_1'(t) & \psi_1'(t) & \gamma_1'(t) \\ \phi_1''(t) & \psi_1''(t) & \gamma_1''(t) \end{array} \right] $$

$$ \alpha_3(t) = \left[ \begin{array}{ccc} \phi_1(t) & \psi_1(t) & \gamma_1(t) \\ \phi_1'(t) & \psi_1'(t) & \gamma_1'(t) \\ \phi_1''(t) & \psi_1''(t) & \gamma_1''(t) \end{array} \right] $$

and

$$ \alpha_4(t) = \left[ \begin{array}{ccc} \phi_1(t) & \psi_1(t) & \chi_1(t) \\ \phi_1'(t) & \psi_1'(t) & \chi_1'(t) \\ \phi_1''(t) & \psi_1''(t) & \chi_1''(t) \end{array} \right] $$

while $d_1$, $d_2$, $d_3$ and $d_4$ are constants.

Calculation of $\phi_1(b), \phi_1'(b)$ and $\phi_1''(b)$ from (4.7) and substitution into (4.6) with the initial conditions (4.3) and (4.4), we can get the constants $d_1$, $d_2$, $d_3$ and $d_4$ as follows:
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\[ d_1 = \frac{1}{W} \left[ - f_2 \psi''_\lambda(b) + \int_a^b \alpha_1(t)f_1(t)dt \right], \]
\[ d_2 = \frac{1}{W} \left[ f_2 \psi''_\lambda(b) + \int_a^b \alpha_2(t)f_1(t)dt \right] \]

and \( d_3 = d_4 = 0 \).

Consequently, we deduce that

\[ \phi_1(x) = \frac{f_2}{W} \left[ \psi'_\lambda(x) - \phi_1(x) \psi''_\lambda(x) \right] + \int_a^b G(x,t,\lambda)f_1(t)dt \]

and

\[ \phi_2 = \phi_1(b) \]

where \( G(x,t,\lambda) \) is the Green's function defined by:

\[ G(x,t,\lambda) = \begin{cases} \psi_\lambda(x) \psi_\lambda(t) + \lambda_1(x) \psi_\lambda(t) & \text{for } \lambda_1 < \lambda < \lambda_1 + \alpha \lambda_1 \\\n\psi_\lambda(x) \psi_\lambda(t) + \lambda_1(x) \psi_\lambda(t) & \text{for } \lambda_1 + \alpha \lambda_1 < \lambda \end{cases} \] (4.9)

The form of equations (4.8) and (4.9) shows that the inverse operator \((\lambda I - A)^{-1}\)

is actually compact; for details of argument of theorem 5 in [8, p.120] can be used.

5. EXPANSION THEOREM.

We now arrive at the problem of expanding an arbitrary function \( f(x) \in H \) for

\( x \in [a,b] \) in terms of the eigenfunctions of (1.1). The results of our investigations

are summarized in the following theorem:

THEOREM 5.1. The operator \( A \) in \( H \) has unbounded set of real eigenvalues of finite

multiplicity, (they have at most multiplicity four), without accumulation points

in \((-\infty, \infty)\) and they can be ordered according to the size, \( 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_4 \)

with \( \lambda_1 = \infty \) as \( i \to \infty \). If the corresponding eigenfunctions \( \phi_1, \phi_2, \phi_3, \ldots \) form a

complete orthonormal system, then for any function \( f(x) \in H \), we have the expansion:

\[ f(x) = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i \]

(4.10)

which is a uniformly convergent series.

The above theorem has some interesting corollaries for particular choices of \( f \).

COROLLARY 4.1. If \( f_1 \in L^2(a,b) \) and \( f = (f_1,0) \in H \), then we have

\[ \phi_1 = \sum_{i=1}^{\infty} \left( \int_a^b \phi_i(x) f_1(x) dx \right) \phi_i(x) \]

\[ \phi_2 = \sum_{i=1}^{\infty} \left( \int_a^b \phi_i(x) f_1(x) dx \right) \phi_{i+1}(x) \]
COROLLARY 4.2. If $0 = (\phi_{11}(x), \phi_{12}) \in D(A)$ and $f = (0, 1) \in H$, we have:

\begin{align*}
(1) \quad 0 = \sum_{i=1}^{\infty} \phi_{12} \phi_{11}(x) = \sum_{i=1}^{\infty} \phi_{11}(b) \phi_{12}(x). \\
(11) \quad 1 = \sum_{i=1}^{\infty} [\phi_{12}^2] = \sum_{i=1}^{\infty} [\phi_{11}(b)^2].
\end{align*}

REFERENCES