SOME APPLICATIONS OF SCHWARZ LEMMA FOR OPERATORS

AKSHAYA KUMAR MISHRA
School of Mathematical Sciences
Sambalpur University
Jyoti Vihar 768019, Orissa
India

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ABSTRACT. A generalized Schwarz lemma and some Harnack type inequalities for operators have been obtained in this paper.

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1. INTRODUCTION.
Let $A$ be a bounded linear operator on a complex Hilbert space $H$. For a complex valued function $f$ analytic on a domain $E$ of the complex plane containing the spectrum $\sigma(A)$ of $A$, let $f(A)$ denote the operator on $H$ defined by the Riesz Dunford integral ([2, p.568]).

$$f(A) = \frac{1}{2\pi i} \int_C f(z) (zI-A)^{-1}dz,$$

where $C$ is a positively oriented simple closed rectifiable contour containing $\sigma(A)$ in its inside domain $G$ and satisfying $C \cup \sigma \subseteq E$. Fan[3] has obtained Schwarz lemma for $f(A)$ and has given several applications of his results including the Harnack's inequalities for operators in [3,4].

In this paper, we obtain a generalized Schwarz lemma and some further Harnack type inequalities for operators.

2. SOME PRELIMINARY LEMMAS.
We need the following lemmas.

LEMMA 1. Let $a,b,c,d$ be complex numbers such that $ad - bc \neq 0$, $c \neq 0$ and let $T$ be a bounded linear operator on a Hilbert space $H$ such that $-d/c$ is not in $\sigma(T)$. Then

$$\| (aT + bI) (cT + dI)^{-1} \| \leq r$$

for $0 < r \leq \frac{1}{|a| |c|}$ if and only if
Equality holds in (2.1) and (2.2) simultaneously.

PROOF. The inequality (2.1) is true if and only if
\[ r^2 I - (cT* + dI)^{-1} (\bar{a}T* + bI) (aT + bI) (cT + dI)^{-1} \geq 0 \]
or
\[ (cT* + dI)^{-1} [r^2 (cT* + dI) (cT + dI) - (\bar{a}T* - bI)(aT + bI)] (cT + dI)^{-1} \geq 0 \]
The operator inside the square brackets can be written as
\[
\begin{align*}
\frac{r^2|ad-bc|}{|a|^2 - r^2|c|^2} I - \{T^*T + \bar{a}b - r^2 \bar{c}d \} T^* + \frac{a\bar{b} - r^2 \bar{c}d}{|a|^2 - r^2|c|^2} T + \frac{|\bar{a}b - r^2 \bar{c}d|^2}{|a|^2 - r^2|c|^2} I \}
\end{align*}
\]
or
\[
\begin{align*}
\frac{r^2|ad-bc|}{|a|^2 - r^2|c|^2} I - \{T^*T + \bar{a}b - r^2 \bar{c}d \} I + \{ T^* + \bar{a}b - r^2 \bar{c}d \} I \}
\end{align*}
\]
This last expression is a positive operator if and only if (2.2) holds. This completes the proof.

**Lemma 2.** Let \(a, b, c, d\) and \(T\) be as in Lemma 1. Then,
\[
\| (aT + bI)(cT + dI)^{-1} - \frac{b\bar{d} - r^2 \bar{a}c}{|d|^2 - r^2|c|^2} I \| \leq \frac{r|ad - bc|}{|d|^2 - r^2|c|^2} \tag{2.3}
\]
for \(0 < r < |d|/|c|\) if and only if \(\|TI\| \leq r\). Equality holds in (2.3) if and only if \(\|TI\| = r\).

PROOF. The inequality (2.3) is equivalent to
\[
\| ((aT + bI)(|d|^2 - r^2|c|^2) - (b\bar{d} - r^2 \bar{a}c)(cT + dI))(cT + dI)^{-1} \| \leq \frac{r|ad - bc|}{|d|^2 - r^2|c|^2} \]
After simplification the above can be written as
\[
\| (dT + r^2 \bar{c}I) (cT + dI)^{-1} \| \leq r \tag{2.4}
\]
Now an application of Lemma 1 shows that (2.4) is equivalent to \(\|TI\| \leq r\).

3. A GENERALIZED SCHWARZ LEMMA.

Let \(D\) denote the open unit disc \(\{z: |z| < 1\}\) in the complex plane and let \(H(D)\)
be the class of complex valued functions analytic in \(D\). Further, let \(B(D) = \{f \in H(D): |f(z)| < 1, z \in D\}\) and let \(B_0(D) = \{f \in B(D): f(0) = 0\}\).
THEOREM 1. Let $f$ be in $B(0)$ and let $A$ be a proper contraction on a Hilbert space $H$. Then,

$$\frac{|A| - |f(0)|}{1-|f(0)|} \leq \frac{|A| + |f(0)|}{1+|f(0)|} .$$

\((3.1)\)

PROOF. Since $f$ is in $B(D)$ and $A$ is a proper contraction, by a result of F and (\cite[Theorem 1, p.276]{3}), $T f(A)$ is also a proper contraction. Now, if we define the complex valued function $g$ by $g(z) = (f(z) - f(0)) (1 - f(0) f(z))^{-1}$ then $g$ is in $B_0(D)$ and $g(A) = (T - f(0) I) (I - f(0) T)^{-1}$ is also a proper contraction. Further, by the operator version of Schwarz lemma (\cite[Corollary 2, p.280]{3}),

$$l g(A) l \leq |A| l .$$

\((3.2)\)

If we take $a = d = 1$, $b = f(0)$, $c = - \overline{f(0)}$ and $r = |A| l$ in Lemma 1 then (3.2) is equivalent to

$$l f(A) - \frac{1-|A|^2}{1-|f(0)|^2} \overline{f(0)} \leq l A l \leq \frac{(1-|f(0)|^2)}{1-|f(0)|^2} l A l .$$

Using triangle inequality we get both the inequalities in (3.1).

COROLLARY 1. Let $f$ in $B_0(D)$ be given by the series $f(z) = b z^n + \ldots$, $b \neq 0$, and let $A$ be a proper contraction on a Hilbert space $H$. Then

$$l A^n l \leq l f(A) l \leq l A^n l (\frac{|A| + |b|}{1+|b|} ) .$$

\((3.3)\)

PROOF. The function $g$, defined by $g(z) = (f(z)/z^n)$, $z \neq 0$ and $g(0) = b$, is in $B(D)$ and $g(A) = A^n g(A)$. Hence the result follows from Theorem 1.

REMARK. The author learned from Professor R. Finn that Theorem 1 follows independently from some results of K. Fan that are now in press.

4. SOME HARNACK TYPE INEQUALITIES.

Let $P(\alpha, \beta)$, $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$, denote the subclass of functions $p$ in $H(D)$ satisfying $p(0) = 1$ and

$$\left| \frac{p(z) - 1}{(2\beta - 1)p(z) + (1 - 2\alpha \beta)} \right| < 1, z \in D .$$

This class of functions have been introduced and studied by Juneja and Mogra \cite{5}. It has been shown in \cite{5} that the $n$th Taylor coefficient $a_n$ of a function $p$ in $P(\alpha, \beta)$ satisfies the sharp inequality $|a_n| \leq 2\beta (1 - \alpha)$. Observe that

$$P(0,1) = \{ p \in H(D): p(0) = 1, \text{Re} \ p(z) > 0, z \in D \} ,$$

$$P(\alpha,1) = \{ p \in H(D): p(0) = 1, \text{Re} \ p(z) > \alpha, z \in D \} ,$$

$$P(0,\beta) = \{ p \in H(D): p(0) = 1, \left| \frac{p(z)}{z} - \frac{1}{2(1-\beta)} \right| < \frac{1}{2(1-\beta)} \} ,$$

$$P(\alpha,\beta) = \{ p \in H(D): p(0) = 1, \left| \frac{p(z)}{z} - \frac{1}{2(1-\beta)} \right| < \frac{1}{2(1-\beta)} \} ,$$
and

\[ P(\alpha, \beta) = P(0,1), \text{ for all admissible choices of } \alpha \text{ and } \beta. \]  

We prove the following theorem which extends a distortion theorem by Kapoor and the author ([6, Theorem 1, p.86]).

**THEOREM 2.** Let \( p \in P(\alpha, \beta) \) be given by the series \( p(z) = 1 + 2b(1-\alpha)z^{\alpha} + \ldots \), \( 0 < |b| \leq 1 \), \( z \in D \) and let \( A \) be a proper contraction on a Hilbert space \( H \). Then,

\[
\| p(A) \| \leq \frac{1 + \| A \| |b| + (1-2\alpha \beta)(\| A \| + |b|)\| A^n \|}{1 + \| A \| |b| + (1-2\beta)(\| A \| + |b|)\| A^n \|},
\]

\[
\| p(A) \| \geq \frac{1 + \| A \| |b| - (1-2\alpha \beta)(\| A \| + |b|)\| A^n \|}{1 + \| A \| |b| - (1-2\beta)(\| A \| + |b|)\| A^n \|},
\]

\[
1 + \| A \| |b| - (1-2\alpha \beta)(\| A \| + |b|)\| A^n \| < \Re p(A),
\]

\[
\| p(A) \| \leq \frac{1 + \| A \| |b| + (1-2\alpha \beta)(\| A \| + |b|)\| A^n \|}{1 + \| A \| |b| + (1-2\beta)(\| A \| + |b|)\| A^n \|},
\]

\[
\Re p(A) \leq \frac{1 + \| A \| |b| + (1-2\alpha \beta)(\| A \| + |b|)\| A^n \|}{1 + \| A \| |b| + (1-2\beta)(\| A \| + |b|)\| A^n \|},
\]

\[
\pm \Im p(A) \leq \frac{2\beta(1-\alpha)(\| A \| + |b|)(1 + \| A \| |b|)\| A^n \|}{(1 + \| A \| |b|)^2 - (1-2\alpha \beta)(\| A \| + |b|)\| A^n \|^2} \leq \Re p(A),
\]

**PROOF.** From the definition of \( P(\alpha, \beta) \), it follows that there exists a function \( w \) in \( B_0(D) \) such that

\[
p(z) = \{1 + (1-2\alpha \beta)w(z)\}^{-1}, \text{ } z \in D
\]

and

\[
p(A) = \{1 + (1-2\alpha \beta)T\}^{-1}, \text{ where } T = w(A).
\]

Further, it is observed that \( w(z) = bz^\alpha \ldots \), \( z \in D \). Hence by Corollary 1., we can say

\[
\| T \| = \| w(A) \| \leq \| A \| (\frac{\| A \| + |b|}{1 + \| A \| |b|}) = r.
\]

Now, choosing \( a = 1-2\alpha \beta, c = 1-2\beta, b = d^{-1} \) in Lemma 2, (4.6) is equivalent to

\[
\| p(A) \| \leq \frac{2r(1-\alpha \beta)}{1-r^2(1-2\beta)^2} \leq \frac{2r(1-\alpha \beta)}{1-r^2(1-2\beta)^2}.
\]

Hence

\[
\| p(A) \| \leq \frac{2r(1-\alpha \beta)}{1-r^2(1-2\beta)^2}.
\]
Substituting the value of \( r \) in the above inequality, we get (4.1) and 4.2.

Also,

\[
\pm \text{Re} [p(A)] - \frac{1-r^2(1-2\beta)(1-2\beta)}{1-(1-2\beta)^2 r^2} I \leq I [p(A) - \frac{1-r^2(1-2\beta)(1-2\beta)}{1-r^2(1-2\beta)^2} I] \leq \frac{2r(1-\alpha)\beta}{1-r^2(1-2\beta)^2} I.
\]

This gives (4.3) and (4.4). Similarly,

\[
\pm \text{Im} [p(A)] = \pm \text{Im} [p(A) - \frac{1-r^2(1-2\alpha\beta)(1-2\beta)}{1-r^2(1-2\beta)^2} I] \leq \frac{2\beta(1-\alpha) r}{1-r^2(1-2\beta)^2} I.
\]

From this (4.5) follows. This completes the proof.

**REMARK.** The right hand side of (4.1) and (4.2) are increasing and decreasing function of \( |b| \), respectively. For the case \( |b| =1, \alpha=0 \) and \( \beta =1 \), our Theorem 2 includes some results of Fan ([3, Corollary 3, p.281], [4, Proposition 2, p.335]).

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**REFERENCES**