BOUNDDED SPIRAL-LIKE FUNCTIONS WITH FIXED SECOND COEFFICIENT

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ABSTRACT. Let $F_{p}(a,\beta, M)$ $(0 < p < 1, |a| < \frac{\pi}{2}, o < \beta < 1$ and $M > \frac{1}{2})$, denote the class of functions $f(z)$ which are regular in $U = \{z:|z| < 1\}$ and of the form

\[ f(z) = z + a_{2} e^{-ia_{z}^{2}} + \ldots, \]

where $|a_{z}| = p(1 + \sigma)(1-\beta) \cos \alpha$, which satisfy for fixed $M$, $z \in U$ and

\[ \left| \frac{e^{-ia z f'(z)} - \beta \cos \alpha - i \sin \alpha}{f(z)} - M \right| < M. \]

In this paper we have found the sharp radius of $\gamma$-spiralness of the functions belonging to the class $F_{p}(a,\beta, M)$.

KEY WORDS AND PHRASES. Spiral-like, bounded functions, radius of $\gamma$-spiralness.

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1. INTRODUCTION. Let $A$ denote the class of functions which are regular and univalent in the unit disc $U = \{z:|z| < 1\}$ and satisfy the conditions $f(0) = 0 = f'(0) - 1$.

Let $F(a,\beta, M) (|a| < \frac{\pi}{2}, 0 < \beta < 1$ and $M > \frac{1}{2})$ denote the class of bounded $a$-spiral-like functions of order $\beta$, that is $f \in F(a,\beta, M)$ if and only if for fixed $M$,

\[ \left| \frac{e^{-ia z f'(z)} - \beta \cos \alpha - i \sin \alpha}{f(z)} - M \right| < M, z \in U. \]  

(1.1)

The class $F(a,\beta, M)$ introduced by Aouf [1], he proved that if $f(z) = z + a_{2} z^{2} + \ldots \in F(a,\beta, M)$ then,

\[ |a_{2}| < (1 + \sigma)(1-\beta) \cos \alpha, \quad \sigma = 1 - \frac{1}{M}. \]  

(1.2)
If \( e = \exp(-i \arg a_2 - i\alpha) \), then

\[
\frac{f(\varepsilon z)}{\varepsilon} = z + a_2 e^{-i\alpha} + \ldots \in F(\alpha, \beta, M),
\]

whenever \( f(z) \in F(\alpha, \beta, M) \). Thus without loss of generality we can replace the second coefficient \( a_2 \) of \( f(z) \in F(\alpha, \beta, M) \) by \( |a_2| e^{-i\alpha} \).

Let \( F_p(\alpha, \beta, M) \) denote the class of functions

\[ f(z) = z + a_2 e^{-i\alpha} + \ldots, \]

which satisfy (1.1), where \( |a_2| = p(l + o)(l - \beta) \cos \alpha \). In view of (1.2) it follows that \( 0 < p < 1 \).

Let \( G_p(\alpha, \beta, M) \) denote the class of functions

\[ g(z) = z + b_2 e^{-i\alpha} + \ldots, \]

regular in \( U \) and satisfy the condition

\[
\left| \frac{e^{-i\alpha + \frac{zg''(z)}{g'(z)}} - \beta \cos - \sin \alpha}{(l - \beta) \cos \alpha} - M \right| < M, \quad z \in U,
\]

where \( |b_2| = \frac{1}{2} p(l + o)(l - \beta) \cos \alpha \).

It follows from (1.1) and (1.3) that

\[ g(z) \in G_p(\alpha, \beta, M), \text{ if and only if } zg''(z) \in F_p(\alpha, \beta, M). \]

We note that by giving specific values to \( p, \alpha, \beta \) and \( M \), we obtain the following important subclasses studied by various authors in earlier papers:

(i) \( F_{1}(\alpha, \beta, M) = F_M(\alpha, \beta) \) and \( G_{1}(\alpha, \beta, M) = G_M(\alpha, \beta) \), are respectively the class of bounded spirallike functions of order \( \beta \) and the class of bounded Robertson functions of order \( \beta \) investigated by Aouf [1] and \( F_{1}(\alpha, 0, M) = F_{\alpha, M} \) and \( G_{1}(\alpha, 0, M) = G_{\alpha, M} \), are respectively the class of bounded spirallike functions and the class of bounded Robertson functions investigated by Kulshrestha [2].

(ii) \( F_p(\alpha, \beta, M) = F(\alpha, \beta) \) and \( G_p(\alpha, \beta, M) = G(\alpha, \beta) \), are considered by Umarani [3].

In this paper we determine the sharp radius of \( \gamma \)-spirallness of the functions belonging to the class \( F_p(\alpha, \beta, M) \), generalizing an earlier result due to Kulshrestha [2], Libera [4], Umarani [5,3].

The technique employed to obtain this result is similar to that used by McCarty [6] and Umarani [3].

2. THE SHARP RADIUS OF \( \gamma \)-SPIRALLNESS OF THE CLASS \( F_p(\alpha, \beta, M), M > 1 \).

**Lemma 1.** If \( f(z) \in F_p(\alpha, \beta, M) \), \( M > 1 \), then

\[
\left| \frac{zf''(z)}{f(z)} - \omega \right| < \rho_0,
\]

(2.1)
where
\[ w_o = \frac{(1+pr)^2 + [(1-\beta)\left(\frac{1+\sigma}{\sigma}\right) -1] \cos \alpha - i \sin \alpha} {1 - pr^2(1+2pr + r^2)} e^{-\frac{t}{r+p}^2} \] (2.2)

and
\[ \rho_o = \frac{(1+\sigma)(1-\beta)\cos \alpha r(1+pr)(r+p)} {1 - pr^2(1+2pr + r^2)} . \] (2.3)

This result is sharp.

PROOF. Let \( f(z) \in F(\alpha, \beta, M), M > 1 \), then there exists a function \( w(z) \) analytic in \( U \) and \( |w(z)| < 1 \) in \( U \) such that
\[ e^{\frac{\alpha}{\sigma} \frac{zf'(z)}{f(z)}} = \cos \alpha \left[ \frac{1 + [(1-\beta)\left(\frac{1+\sigma}{\sigma}\right) -1] \sigma w(z)} {1 - \sigma w(z)} \right] + i \sin \alpha, \sigma = 1 - \frac{1}{M} \]
or
\[ \frac{zf'(z)}{f(z)} = \frac{1 + [(1-\beta)\left(\frac{1+\sigma}{\sigma}\right) -1] \cos \alpha - i \sin \alpha} {1 - \sigma w(z)} e^{-\frac{t}{r+p}^2} \]

Solving for \( w(z) \),
\[ w(z) = \frac{zf'(z) - 1} {\sigma [zf'(z) + [(1-\beta)\left(\frac{1+\sigma}{\sigma}\right) -1] \cos \alpha - i \sin \alpha] e^{-\frac{t}{r+p}^2}} . \]

Since \( f(z) = z + |a_2|e^{-\frac{t}{r+p}^2} + \ldots \), we obtain \( w(z) = pz + \ldots = z\phi(z) \), where \( \phi(z) \)
is analytic in \( U \), \( \phi(0) = p \) and \( |\phi(z)| < 1 \) in \( U \). Now \( \frac{\phi(z) - p}{1 - p\phi(z)} \) \( z \). Therefore
\[ \phi(z) = \frac{z + p} {1 + pz} . \]

Also \( |w(z)| = |z\phi(z)| < \frac{|z| + p} {1 + |z|p} |z| . \) Let \( g(z) = \frac{|z| + p} {1 + p|z|} z \)
and
\[ h(z) = \frac{1 + [(1-\beta)\left(\frac{1+\sigma}{\sigma}\right) -1] \cos \alpha - i \sin \alpha} {1 - \sigma z} e^{-\frac{t}{r+p}^2} \]

Since the image of \( |z| < r \) under \( g(z) \) is a disc and \( h(z) \) is a bilinear transformation, then \( \frac{zf'(z)}{f(z)} \) is subordinate to \( (hog)(z) \). That is, the image of \( |z| < r \) under \( \frac{zf'(z)}{f(z)} \) is contained in the image of \( |z| < r \) under \( (hog)(z) \).

Equality in (2.1) can be attained by a function
\[ f(z) = z(1-2\sigma z + \sigma^2)^{-\frac{1+\sigma}{2\sigma}(1-\beta)\cos \alpha} e^{-\frac{t}{r+p}^2} \] (2.4)
hence

\[
\frac{zf''(z)}{f(z)} = \frac{1 - 2p \sigma z + \sigma^2 - (1+\sigma)(1-\beta)\cos \alpha z(z-p)}{1 - 2p \sigma z + \sigma^2} = \frac{1 + \sigma \psi(1+\sigma)(1-\beta)\cos \alpha e^{-i\psi}}{1 + \sigma \psi},
\]

where \( \psi = \frac{z(z-p)}{1 - \rho \sigma z} \).

Since \( p < 1, 0 < \sigma < 1, |\psi| < 1 \) for \( z \in U \).

This shows that

\[
e^{i\alpha} \frac{zf''(z)}{f(z)} = \frac{1}{1 + \sigma \psi(z)} \left[ 1 + \frac{(1+\sigma)(1-\beta)\cos \alpha z}{1 + \sigma \psi(z)} \right] + i \sin \alpha
\]

and

\[
e^{i\alpha} \frac{zf''(z)}{f(z)} - i \sin(1-\beta)\cos \alpha
\]

\[
= \frac{1 - \psi(z)}{1 + \sigma \psi(z)}. \]

Then it is easy to show that \( \frac{|1 - \psi(z)|}{1 + \sigma \psi(z)} < M \), \( \sigma = 1 - \frac{1}{M} \). Thus \( f \in F_{p}(a, \beta, M) \).

Substituting \( \psi = -\frac{\delta(\delta - \sigma e^{i\alpha})}{\sigma(1-\sigma e^{i\alpha})} \), where \( \delta = \frac{f(r+p)}{1+r \rho} \) in (2.5), we find that

\[
\frac{|zf''(z)|}{|f(z)|} - \omega_0 \left| p \rho_0 \right|, \text{ where } \omega_0 \text{ and } \rho_0 \text{ are given by (2.2) and (2.3)}.
\]

This completes the proof of the lemma.

REMARK 1.

(i) If \( p=1 \) and \( \beta=0 \) in Lemma 1, we obtain a result of Kulshrestha [2].

(ii) If \( M = \omega(\sigma=1) \) in Lemma 1, we obtain a result of Umarani [3].

(iii) If \( \alpha=0 \) and \( M = \omega(\sigma=1) \) in Lemma 1, we obtain a result of McCarty [6].

THEOREM 1. If \( f(z) \in F_{p}(a, \beta, M) \), \( r > 1 \), then \( f(z) \) is \( \gamma \)-spiral

\( |z| < r_\gamma \), where \( r_\gamma \) is the smallest positive root of the equation

\[
\cos \gamma + p \left[ 2 \cos \gamma - (1+\sigma)(1-\beta)\cos \alpha \right] r + \frac{[p^2 \cos \gamma + \sigma^2 - (1+\sigma)(1-\beta)\cos \alpha(1+ \rho^2)] r^2}{[2c-(1+\sigma)(1-\beta)\cos \alpha] r^3 + c r^4} = 0,
\]

(2.6)
where \( c = \cos(\gamma - 2\alpha) + [(1 - \beta)\left(\frac{1+\sigma}{\sigma}\right) - 2] \cos \alpha \cos(\gamma - \alpha). \) The result is sharp.

**Proof.** Let \( f(z) \in \mathcal{F}(\alpha, \beta, M), M > 1, \) then by the above Lemma, we have

\[
\left| \frac{zf'(z)}{f(z)} - w_0 \right| < \rho_0.
\]

Hence \( \Re e^{iy} \frac{zf'(z)}{f(z)} > \Re e^{iy}. w_0 - \rho_0 \)

\[
cos \gamma (1+pr)^2 + \Re \left\{ [(1 - \beta)\left(\frac{1+\sigma}{\sigma}\right) - 1] \cos \alpha \sin \alpha \right\} e^{i(\gamma - \alpha)} r^2(r+p)^2
\]

\[
= \frac{-(1+\sigma)(1-\beta)\cos \alpha(1+pr)(r+p)}{(1 - r^2)(1 + 2pr + r^2)}.
\]

\[
f(z) \text{ is } \gamma\text{-spiral if the R.H.S. of (2.7) is positive. Hence } f(z) \text{ is } \gamma\text{-spiral for}
\]

\[
|z| < r_y \quad \text{where } r_y \text{ is the smallest positive root of the equation}
\]

\[
cos \gamma (1+pr)^2 + \{\cos(\gamma - 2\alpha) + [(1 - \beta)\left(\frac{1+\sigma}{\sigma}\right) - 2] \cos \alpha \cos(\gamma - \alpha)\} r^2(r+p)^2
\]

\[
= \frac{-(1+\sigma)(1-\beta)\cos \alpha(1+pr)(r+p)}{(1 - r^2)(1 + 2pr + r^2)}. \tag{2.7}
\]

\( f(z) \) is \( \gamma\)-spiral if the R.H.S. of (2.7) is positive. Hence \( f(z) \) is \( \gamma\)-spiral for

\[
|z| < r_y \quad \text{where } r_y \text{ is the smallest positive root of the equation}
\]

\[
cos \gamma (1+pr)^2 + \{\cos(\gamma - 2\alpha) + [(1 - \beta)\left(\frac{1+\sigma}{\sigma}\right) - 2] \cos \alpha \cos(\gamma - \alpha)\} r^2(r+p)^2
\]

\[
-(1+\sigma)(1-\beta)\cos \alpha(1+pr)(r+p) = 0.
\]

Simplifying the above equation, we obtain (2.6).

If \( \gamma = 0 \) in the above theorem, we obtain the radius of starlikeness of the class

\[
\mathcal{F}(\alpha, \beta, M).
\]

**Corollary 1.** \( f(z) \in \mathcal{F}(\alpha, \beta, M), M > 1, \) is starlike for \( |z| < r_0, \) where \( r_0 \) is the least positive root of the equation

\[
1+p [2-(1+\sigma)(1-\beta)\cos \alpha] r + 
\]

\[
\left(\frac{1+\sigma}{\sigma}\right)(1-\beta) \cos \alpha [\cos \alpha^2 - \sigma (1+p^2)] r^2 + 
\]

\[
p[2c-(1+\sigma)(1-\beta)\cos \alpha] r^3 + cr^4 = 0, \tag{2.8}
\]

where \( c = (\frac{1+\sigma}{\sigma}) (1-\beta) \cos^2 \alpha - 1. \)

If \( p=1, \gamma=0 \) and \( \beta=0 \) in Theorem 1, we obtain a result of Kulshrestha [2].

**Corollary 2.** \( f(z) \in \mathcal{F}(\alpha, M), M > 1, \) is starlike for \( |z| < r_0, \) where \( r_0 \) is the least positive root of the equation

\[
1-(1+\sigma) \cos \alpha r+(\frac{1+\sigma}{\sigma}) \cos^2 \alpha - 1 \] \( r^2 = 0. \)
REMARK 2.

(i) If $M = (\sigma = 1)$ in Theorem 1, we obtain a result of Umarani [3].

(ii) If $p = 1$ and $M = (\sigma = 1)$ in Theorem 1, we obtain a result of Libera [4] and Umarani [5].

(iii) If $p = 1$, $\beta = 0$, $\gamma = 0$ and $M = (\sigma = 1)$ in Theorem 1, we obtain a result of Robertson [7].

Since $g(z) \in G_{p}(\alpha, \beta, M)$ if and only if $z^{p}g(z)^{n} \in F_{p}(\alpha, \beta, M)$ we obtain from Theorem 1,

\[ \text{THEOREM 2. } \text{If } g(z) \in G_{p}(\alpha, \beta, M), M > 1, \text{ then } \Re \left( e^{\frac{i\gamma}{M}}(1 + \frac{z^{p}g(z)}{g'(z)}) \right) > 0 \text{ for } |z| < r_{\gamma}, \text{ where } r_{\gamma} \text{ is the least positive root of equation (2.6).} \]

The result is sharp.

If $\gamma = 0$ in Theorem 2, we obtain the radius of convexity of the class $G_{p}(\alpha, \beta, M)$.

COROLLARY 3. If $g(z) \in G_{p}(\alpha, \beta, M), M > 1$, then the radius of convexity of $g(z)$ is the least positive root of equation (2.8).

REMARK 3.

(i) For $M = (\sigma = 1)$ in Theorem 2, and Corollary 3, we obtain a result of Umarani [3].

(ii) If $p = 1$ and $\beta = 0$ in Corollary 3, we obtain a result of Kulshrestha [2].

(iii) For $p = 1$ and $M = (\sigma = 1)$, Theorem 2, generalizes the result of Umarani [5].

REFERENCES

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