ABSTRACT. Consequences of the existence of conformal vector fields in (locally) symmetric and conformal symmetric spaces, have been obtained. An attempt has been made for a physical interpretation of the consequences in the framework of general relativity.

KEY WORDS AND PHRASES: Symmetric spaces, Conformal symmetric spaces, Conformal vector field, Null Killing vector field, Killing horizon.


1. INTRODUCTION.

Let $M$ denote a semi-Riemannian manifold with metric tensor $g_{ab}$ of arbitrary signature. All the geometric objects defined on $M$ are assumed sufficiently smooth. Although our treatment is local, nevertheless we shall drop the term 'locally', for example in 'locally symmetric'. We denote the Christoffel symbols by $\Gamma_{bc}^a$ and the covariant differentiation by a semi-colon $;$. We say that $M$ is symmetric in Cartan's sense if the Riemann curvature tensor $R_{bcd}^a$ is covariant constant, i.e. $R_{bcd;e}^a = 0$. We say that $M$ is conformal symmetric [1] if its Weyl conformal curvature tensor $C_{bcd}^a$ is covariant constant, i.e. $C_{bcd;e}^a = 0$. Thus a symmetric space is conformal symmetric but the converse is not necessarily true.

A vector field $\xi$ on $M$ is said to be conformal if

$$L_\xi g_{ab} = 2\sigma g_{ab} \tag{1.1}$$

where $L_\xi$ denotes the Lie-derivative operator via $\xi$ and $\sigma$ denotes a scalar function on $M$. In particular, if $\sigma = 0$, $\xi$ is called a Killing vector field and if $\sigma$ is a non-zero constant, $\xi$ is called a homothetic vector field. It is known that a conformal vector field $\xi$ satisfies:

$$L_\xi C_{bcd}^a = 0 \tag{1.2}$$

Equation (1.1) implies

$$L_\xi r_{ab}^c = \sigma_a \sigma_b + \sigma_b \sigma_a - g_{ab} g_{cd} \sigma^d \tag{1.3}$$

but the converse is not necessarily true. However, we know [2] that (1.3) is
equivalent to
\[ L_\tau g_{ab} = 2\sigma g_{ab} + h_{ab} \] (1.4)
where \( h_{ab} \) is a covariant constant tensor field. A vector field \( \tau \) satisfying (1.3) or (1.4) is said to be affine conformal [2] and is said to generate a one-parameter group of conformal collineations [3,4]. An affine conformal vector field with constant \( \sigma \) (i.e., \( L_\tau R^a_{bcd} = 0 \)) is known as an affine Killing vector field (which preserves the geodesics). For (i) a compact orientable positive definite Riemannian manifold without boundary, (ii) an irreducible positive definite Riemannian manifold and (iii) an \( n(n > 2) \) - dimensional non-flat space-form; an affine conformal vector field reduces to conformal vector field. For a non-Einstein conformally flat space of dimension \( > 2 \); Levine and Katzin [5] proved that \( h_{ab} \) is a linear combination of \( g_{ab} \) and the Ricci tensor \( R_{ab} \).

Conformal motion (generated by a conformal vector field) is a natural symmetry of the space-time manifolds in general relativity, inherited by its causality-preserving [6] character. But sometimes, it is desirable to consider conformal motions which provide covariant conservation law generators. It was pointed out by Katzin et al [7] that there is a fundamental symmetry called curvature collineation (CC) defined by a vector field \( \tau \) satisfying
\[ L_\tau R^a_{bcd} = 0 \] (1.5)
Komar's identities [8] (which define a conservation law generator) follow naturally by the existence of CC. A conformal vector which also generate CC, is called a special conformal vector. A conformal vector is special conformal vector iff
\[ \sigma_{ab} = 0 \] (1.6)
The purpose of this paper is to study the consequences of the existence of (i) a special conformal vector field in a symmetric space and (ii) a conformal vector field in a conformal symmetric space; and indicate the physical interpretation of the consequences within the framework of general relativity.

2. SYMMETRIC AND CONFORMAL SYMMETRIC SPACES.

Here we prove two theorems as follows:

**THEOREM I.** Let a non-flat symmetric space \( M \) of dimension \( n > 4 \), admit a special conformal vector field \( \tau \). Then either (i) \( M \) has zero scalar curvature and \( \nabla \sigma \) is a null Killing vector field, or (ii) \( \tau \) reduces to a homothetic vector field.

(Note that the above theorem is valid also for an affine conformal vector field, in which case the alternative conclusion (ii) would be: \( \tau \) reduces to an affine Killing vector field. The proof is common).

**PROOF.** We have the following identity [9]:
\[
L_\tau (R^b_{cde};a) - (L_\tau R^b_{cde};a) = (L_\tau R^f_{aef}) R^b_{cde} - (L_\tau R^f_{aef}) R^b_{cde} - (L_\tau R^f_{aef}) R^b_{cde} - (L_\tau R^f_{aef}) R^b_{cde}
\]
By our hypothesis the left hand side vanishes and consequently, in view of (1.3) the above equation assumes the form:
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\[ \frac{\sigma_a}{\sigma_f} R^f_{\text{cde}} + \sigma_a R^b_{\text{cde}} - \sigma^b R^b_{\text{acde}} = \sigma_{c} R^b_{\text{ade}} \]

\[ + \sigma_a R^b_{\text{cde}} - g_{ab} R^b_{\text{fde}} \sigma^f + \sigma_d R^b_{\text{ca}} + \sigma_a R^b_{\text{cde}} \]

\[ - g_{ad} R^b_{\text{cfe}} \sigma^f + \sigma_a R^b_{\text{cde}} + \sigma_a R^b_{\text{cde}} - g_{ae} R^b_{\text{cdf}} \sigma^f \]

(2.1)

where \( \sigma_a \) stands for \( \sigma_{;a} \). Taking the product of both sides with \( \sigma^a \) yields

\[ (\sigma_a \sigma^a) R^b_{\text{cde}} = 0 \]

As per our hypothesis, \( M \) is not flat and therefore the above equation shows

\[ \sigma_a \sigma^a = 0 \]

which implies that either (i) \( \text{grad} \sigma \) is null (a non-zero vector of zero norm), or (ii) \( \sigma \) is constant. We first take up case (i). Successive contractions of (2.1) lead to

\[ (n - 4) \sigma_f R^f_{\text{e}} = R \sigma_e \]

(2.2)

Two subcases arise: If \( n = 4 \), then (2.2) gives \( R = 0 \). If \( n > 4 \), then using the condition (1.6) obtains \( R^f_{\text{e}} \sigma^f = 0 \). This, substituted in (2.2), shows that \( R = 0 \). Thus, in case (i) \( \text{grad} \sigma \) is null and Killing (in virtue of (1.6)) and the scalar curvature \( R \) vanishes identically. In case (ii) \( \xi \) is homothetic or affine Killing according as \( \xi \) is conformal or affine conformal. This proves the theorem.

**THEOREM 2.** Let a conformal symmetric space \( M \) (dim \( M > 3 \)) admit a conformal vector field \( \xi \). Then one of the following holds:

(i) \( M \) is conformally flat

(ii) \( \text{grad} \sigma \) is a null vector

(iii) \( \xi \) reduces to a homothetic vector field.

In particular, if \( \xi \) were non-homothetic special conformal vector field and \( M \) were not conformally flat, then \( \text{grad} \sigma \) would have been null and Killing too.

**PROOF.** Consider the identity [9]:

\[ L_\xi (C^b_{\text{cde}}; a) - (L_\xi C^b_{\text{cde}}); a = (L_\xi \Gamma^f_{\text{af}})C^f_{\text{cde}} - (L_\xi \Gamma^f_{\text{af}})C^f_{\text{cde}} \]

\[ - (L_\xi \Gamma^f_{\text{af}})C^b_{\text{cfe}} - (L_\xi \Gamma^f_{\text{af}})C^b_{\text{cfe}} \]

Observe that the left hand side vanishes because \( M \) is conformal symmetric and (1.2) holds for a conformal vector field. By use of (1.3) in the above equation and contracting at \( a \) and \( b \) we obtain (noting dim \( M > 3 \))

\[ \sigma^a \sigma_a C^b_{\text{cde}} = 0 \]

Therefore we conclude that either (i) \( C^b_{\text{cde}} = 0 \) meaning \( M \) is conformally flat, or \( \sigma^a \sigma_a = 0 \) so that (ii) \( \sigma \) is a null vector or (iii) \( \sigma \) is constant. Thus we have proved that one of the following is true: (i) \( M \) is conformally flat, (ii) \( \text{grad} \sigma \) is a
null vector, (iii) $\xi$ is homothetic. In particular, if $\xi$ were non-homothetic special conformal vector field and $M$ not conformally flat then, of course, (ii) holds. Moreover, in this case $\nabla \sigma$ would be Killing in virtue of the condition (1.6) for special conformal vector field. This completes the proof.

**Remark 1.** The conclusion (i) of Theorem 2 can be highlighted by saying that, if a conformal symmetric space $M$ admits a one-parameter group of conformal motions (such that $\nabla \sigma$ is neither null nor zero) then $M$ is conformally flat. This can be compared with the standard result: If an $n$-dimensional semi-Riemannian manifold $M$ admits a maximal, i.e. $\frac{1}{2}(n+1)(n+2)$-parameter group of conformal motions, then $M$ is conformally flat.

**Remark 2.** The conclusion (i) of Theorem 1 can be interpreted in the context of general relativity as follows. Let $M$ be the space-time manifold of general relativity and satisfy the hypothesis of Theorem 1. $M$ with zero scalar curvature, is a space-time carrying pure radiation [10] (e.g. massless scalar fields, neutrino fields or high frequency gravitational waves) and Einstein-Maxwell field. $M$ with the gradient of conformal scalar field as a null Killing field, has a Killing horizon [11] defined by the null hypersurfaces of transitivity, $\sigma = \text{constant}$.

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**References**


