A CHARACTERIZATION OF THE ALGEBRA OF HOLomorphic FUNCTIONS ON A SIMPLY CONNECTED DOMAIN

DERMING WANG and SALEEM WATSON
Department of Mathematics and Computer Science
California State University, Long Beach
Long Beach, California 90840

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ABSTRACT: Let A be a singly-generated $\mathcal{F}$-algebra. It is shown that A is isomorphic to $\mathcal{H}(\Omega)$ where $\Omega$ is a simply connected domain in $\mathbb{C}$ if and only if A has no topological divisors of zero. It follows from this that there are exactly three $\mathcal{F}$-algebras (up to isomorphism) which are singly generated and have no topological divisors of zero.

KEY WORDS AND PHRASES. $\mathcal{F}$-algebras, holomorphic functions, topological divisors of zero

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1. INTRODUCTION.

The algebra $\mathcal{H}(\Omega)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ with pointwise operations and compact-open topology is an interesting example of an $\mathcal{F}$-algebra. This algebra has been characterized in terms of some of the special properties it enjoys that are derived from the fact that it consists of holomorphic functions. (See for example [1], [2], [3], [4] and [5] for characterizations in terms of the local maximum modulus principle, the Cauchy estimate, Montel's theorem, the existence of derivations, and Taylor's theorem.) In [6] a characterization of the algebra of entire functions in terms of Liouville's theorem is given.

Watson [5] shows that an $\mathcal{F}$-algebra A which has a Schauder basis that is generated by an element $z \in A$ with open spectrum is algebraically and topologically isomorphic to $\mathcal{H}(\Omega)$ where $\Omega$ is an open disk in $\mathbb{C}$. In this paper we study $\mathcal{F}$-algebras that are generated by a single element $z$ (without requiring that $z$ generate a basis for A). Of course, this condition alone is not enough to completely describe the algebra $\mathcal{H}(\Omega)$ among $\mathcal{F}$-algebras. We will show, however, that this together with the condition that A has no topological divisors of zero, completely characterizes $\mathcal{H}(\Omega)$ for a simply connected domain $\Omega$. It follows from this that there are exactly three singly generated $\mathcal{F}$-algebras (up to isomorphism) which have no topological divisors of zero.

2. PRELIMINARIES.

An $\mathcal{F}$-algebra is a complete metrizable locally $m$-convex algebra. (All the algebras we consider are assumed to be commutative algebras over $\mathbb{C}$.) The topology of such an algebra is given by an increasing sequence of seminorms $\{p_n|n \in \mathbb{N}\}$. Each $p_n$ determines a Banach algebra $A_n$ which is the completion of $A/\ker(p_n)$. If $n \leq m$ then the natural homomorphism...
from $A/\ker(p_m)$ to $A/\ker(p_n)$ induces a norm decreasing homomorphism $\tau_{nm}: A_m \to A_n$ whose range is a dense subalgebra of $A_n$. The Banach algebras $A_n$ with maps $\tau_{nm}$ form an inverse limit system and $\varprojlim (A_n, \tau_{nm})$ is topologically and algebraically isomorphic to $A$.

The maximal ideal space of $A$ is the space $\hat{\mathcal{M}}(A)$ consisting of all non-zero continuous multiplicative linear functionals on $A$ endowed with the Gelfand topology. This topology is the weak topology on $\hat{\mathcal{M}}(A)$ generated by the Gelfand transforms $\hat{x}: \hat{\mathcal{M}}(A) \to \mathbb{C}$ defined by $\hat{x}(f) = f(x)$. The map $\gamma: A \to \hat{A}$ is a continuous homomorphism onto the algebra $\hat{A} \subseteq C(\hat{\mathcal{M}}(A))$ of Gelfand transforms. For each $n \in \mathbb{N}$ the quotient map $\tau_n$ from $A$ onto $A/\ker(p_n)$ induces a homeomorphism $\tau_n^*\hat{\mathcal{M}}(A_n)$ of the maximal ideal space $\hat{\mathcal{M}}(A_n)$ of $A_n$ onto a compact subset $M_n$ of $\hat{\mathcal{M}}(A)$. For $n < m$ we have $M_n \subseteq M_m$ and $\mathcal{M}(A) = \bigcup M_n$.

The spectrum of $z \in A$ is the set $\sigma(z) = \{f(z) \mid f \in \mathcal{M}(A)\}$. For each $n \in \mathbb{N}$ the set $\sigma_n(z) = \{f(z) \mid f \in M_n\}$ and $\sigma = \bigcup \sigma_n$. The element $z \in A$ generates $A$ if $A$ is the smallest closed subalgebra containing $z$ and $e$ (the identity of $A$). In this case the spectrum map $\varphi: \mathcal{M}(A) \to \sigma(z)$ defined by $f \mapsto f(z)$ is a continuous bijection.

3. CHARACTERIZING

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. The algebra $H(\Omega)$ of holomorphic functions on $\Omega$ is an $\mathcal{F}$-algebra in the compact-open topology. It is well known that $H(\Omega)$ has no (nonzero) topological divisors of zero [9], and is singly-generated. We will show that these last two properties of $H(\Omega)$ completely characterize it among $\mathcal{F}$-algebras.

For the rest of this paper $A$ will denote an $\mathcal{F}$-algebra with identity $e$ which is generated by $z$, where $z$ is not a scalar multiple of $e$, and which has no nonzero topological divisors of zero.

**LEMMA 1.** $A$ is semisimple and so the Gelfand transform is a bijection.

**PROOF:** Suppose $y \in \text{Rad}(A)$, $y \neq 0$. Then $\sigma(y) = \{0\}$ and by [8, Proposition 11.8] $y$ is a topological divisor of zero.

**LEMMA 2.** The spectrum $\sigma(z)$ is a domain in $\mathbb{C}$.

**PROOF:** If $\lambda \in \sigma(z)$ is a boundary point of $\sigma(z)$, then again by [8, Proposition 11.8], $z - \lambda e$ is a topological divisor of zero. Thus $\sigma(z)$ is open.

If $\sigma(z)$ includes the two components $U_1$ and $U_2$, then the characteristic functions $h_1$ of $U_1$ and $h_2$ of $U_2$ are analytic on $\sigma(z)$. By the functional calculus there exist $x_1, x_2 \in A$ with $\tilde{x}_1 = h_1(\tilde{z})$ and $\tilde{x}_2 = h_2(\tilde{z})$. Clearly $\tilde{x}_1 \tilde{x}_2 = 0$ so by Lemma 1 $x_1 x_2 = 0$ and thus these elements are nonzero (topological) divisors of zero.

**LEMMA 3.** The domain $\sigma(z)$ is simply connected.

**PROOF:** Let $\varphi: \mathcal{M}(A) \to \sigma(z)$ be the spectrum map and for $t \in \sigma(z)$ we use the notation $f_t = \varphi^{-1}(t)$. For each $x \in A$ define $\tilde{x}: \sigma(z) \to \mathbb{C}$ by $\tilde{x}(t) = \tilde{x}(\varphi^{-1}(t)) = \varphi(f_t)$. We show that $\tilde{x}$ is analytic on $\sigma(z)$. Since $A$ is generated by $z$ there exists a sequence of polynomials $p_n(z)$ converging to $x$ in $A$ and so $f(p_n(z)) \to f(x)$, for every $f \in \mathcal{M}(A)$. For $t \in \sigma(z)$, $p_n(t) = p_n(f^{-1}_t) \to f(x)$, so each $\tilde{x}$ is a pointwise limit of polynomials on $\sigma(z)$. We now show that this convergence is uniform on
compact subsets of \( \sigma(z) \) and so each \( \bar{x} \) is analytic on this spectrum. Since \( A \) has no topological divisors of zero, for each \( n \in \mathbb{N} \) there exists \( m \) such that \( \sigma_n \subseteq \text{int} \sigma_m \) (see Arens [9]). So without loss of generality, we may assume that for \( n = 1, 2, \ldots \),

\[
\sigma \subseteq \text{int} \sigma_{n+1} \ldots \sigma_{n+1} \quad \text{and} \quad \sigma = \bigcup \text{int} \sigma_n = \sigma.
\]

Since \( \varphi | M_n \) is a homeomorphism onto its image \( \varphi(M_n) = \sigma_n \), it follows that if \( K \) is a compact subset of \( \sigma \) there exists \( n \in \mathbb{N} \) such that \( K \subseteq \text{int} \sigma_n \subseteq \sigma_n \). Thus \( \varphi^{-1}(K) \subseteq \varphi^{-1}(\sigma_n) = M_n \) and so \( \varphi^{-1}(K) \) is a compact subset of \( \text{M}(A) \). Now \( p_n \to x \) by the continuity of the Gelfand map \( \gamma \), \( \bar{p}_n \to \bar{x} \) in \( \hat{A} \), i.e., the convergence is uniform on compact subsets of \( \text{M}(A) \). Thus for \( \epsilon > 0 \) and sufficiently large \( n \),

\[
|p_n(f_t) - \bar{x}(f_t)| < \epsilon
\]

for \( f_t \in \varphi^{-1}(K) \), which is the same as

\[
|p_n(t) - \bar{x}(t)| < \epsilon
\]

for \( t \in K \). Thus each \( \bar{x} \) is the limit of polynomials, uniformly on compact subsets of \( \sigma(z) \), and hence is analytic there.

Let \( h \in H(\sigma(z)) \). We show that \( h \) is the limit of polynomials in \( H(\sigma(z)) \), then it follows that \( \sigma(z) \) is simply connected. Using the functional calculus for \( \mathcal{F} \)-algebras we find \( x \in A \) such that \( \bar{x}(f) = h(\bar{x}(f)), f \in \text{M}(A) \). Therefore \( h = \bar{x} \). This together with the preceding paragraph completes the proof.

**Lemma 4.** \( \sigma(z) \) is homeomorphic with \( \text{M}(\lambda) \).

**Proof:** The map \( \varphi \) is a continuous bijection. But \( \bar{x} \circ \varphi^{-1} = \bar{x} \) is continuous for each \( \bar{x} \in \hat{A} \) and so the continuity of \( \varphi^{-1} \) follows from the fact that the topology of \( \text{M}(A) \) is the weak topology generated by \( \hat{A} \).

Lemma 4 may also be derived from [7, Theorem 1.3]. Notice that Lemmas 3 and 4 imply that \( \text{M}(A) \) is homeomorphic to the open unit disc. We now prove our main result.

**Theorem 1.** An \( \mathcal{F} \)-algebra \( A \) is algebraically and topologically isomorphic to \( H(\Omega) \) for a simply connected domain \( \Omega \) if and only if \( A \) is singly-generated and has no nonzero topological divisors of zero.

**Proof:** That the \( \mathcal{F} \)-algebra \( H(\Omega) \) has these properties is discussed at the beginning of this section.

Conversely, let \( \hat{A} = (\bar{x} | x \in A) \) and equip \( \hat{A} \) with the compact open topology. From the proof of Lemma 3, \( \hat{A} = H(\sigma(z)) \) algebraically and topologically. Also, \( \hat{A} \) and \( \hat{A} \) are isomorphic as \( \mathcal{F} \)-algebras via the map \( \delta: \hat{A} \to \hat{A} \) by \( \bar{x} \to \bar{x} = \bar{x} \circ \varphi^{-1} \). Since the Gelfand map \( \gamma: \hat{A} \to \hat{A} \) is bijective by Lemma 1, it follows that the map \( \delta \circ \gamma \) is a continuous bijection of \( \hat{A} \) onto \( \hat{A} = H(\sigma(z)) \). The open mapping theorem now yields the result.

The notion of topological divisor of zero we used above is that due to Michael [8, p. 47]. Our Theorem 1 does not remain valid if that notion is replaced by the stronger definition of Arens [10] (called *strong topological divisor of zero* by Michael). In fact, the \( \mathcal{F} \)-algebra \( \mathcal{C}[X] \) of formal power series (with the topology of pointwise convergence in the coefficients) is singly generated and has no strong topological divisors of zero [11]. But this algebra is not isomorphic to \( H(\Omega) \) for any domain \( \Omega \).

The Riemann mapping theorem yields the following corollary:

**Corollary 1.** There are (up to isomorphism) exactly three \( \mathcal{F} \)-algebras which are singly generated and have no nonzero topological divisors of zero. Namely, \( \mathbb{C} \), the algebra \( H(D) \) where \( D \) is the open unit disk, and the algebra \( \mathcal{E} \) of entire functions.
Birtel [6] (see also [12] and [13]) gave a characterization of the algebra of entire functions as a singly-generated Liouville algebra without topological divisors of zero. A Liouville algebra is an \( \mathcal{F} \)-algebra in which every element with bounded spectrum is a scalar multiple of the identity. We give another proof of Birtel's theorem based on our Theorem 1.

**THEOREM 2.** (Birtel) An \( \mathcal{F} \)-algebra \( A \) is topologically and algebraically isomorphic to the algebra \( \mathcal{E} \) of entire functions if and only if \( A \) is a singly-generated Liouville algebra with no nonzero topological divisors of zero.

**PROOF:** By Theorem 1, \( \sigma(z) \) is simply connected and \( A \) is isomorphic to \( H(\sigma(z)) \). If \( \sigma(z) \neq \mathbb{C} \) then there is a one-to-one analytic function \( \psi \) from \( \sigma(z) \) onto \( \mathbb{D} \). Thus there exists \( x \in A \) such that \( \tilde{x} = \psi \circ \tilde{z} \). Clearly \( x \) is not a scalar multiple of \( e \) and \( \sigma(x) = \mathbb{D} \), contradicting the assumption that \( A \) is Liouville.

A natural extension of the notion of a simply connected domain to \( \mathbb{C}^n \) is that of a Runge domain. If \( \Omega \) is a Runge domain in \( \mathbb{C}^n \) then \( H(\Omega) \) is \( n \)-generated and has no nonzero topological divisors of zero. We pose the question of whether a finitely-generated \( \mathcal{F} \)-algebra \( A \) with no nonzero topological divisors of zero is isomorphic to \( H(\Omega) \) for a Runge domain \( \Omega \). In the case that \( A \) has a finitely-generated Schauder basis in which the joint spectrum of the generators is an open set in \( \mathbb{C}^n \), it is shown in [14] that \( A \) is isomorphic to \( H(\Omega) \) for a complete logarithmically convex Reinhardt domain \( \Omega \).

**REFERENCES**