ON SOME FIXED POINT THEOREMS

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ABSTRACT. In this paper we prove a fixed point theorem for inward mappings using a well-known result of Ky Fan type in Hilbert space setting.

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The following well known theorem of Ky Fan has been of great importance in nonlinear analysis, minimax theory and approximation theory [1].

Let $C$ be a nonempty compact, convex subset of a normed linear space $X$ and let $f : C \rightarrow X$ be a continuous mapping. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C),$$

where $d(a, B) = \inf\{\|a - b\| / b \in B\}.$

If $fy \in C$, then $f$ has a fixed point.

There have appeared several extensions of Ky Fan theorem. Lin [2] proved an interesting result for densifying mappings. Reich [3] relaxed compactness and proved the result for approximately compact, convex sets. Other results are due to Sehgal [4], Sehgal and Singh [5], Kapoor [6] and Singh and Watson [7].

In the present paper we prove a fixed point theorem for inward mappings using a result of Ky Fan type theorem for Hilbert space.

For definitions and notations we refer to Browder [8]. We will use his results for our theorem.
Let \( C \) be a closed, bounded, convex subset of \( H \), a Hilbert space. A function \( f: C \to H \) is called semicontractive if there exists a mapping \( T: H \times H \to C \) such that

i) \( f(x) = T(x, x) \) for \( x \in C \), while

ii) for fixed \( x \in H \), \( T(\cdot, x) \) is nonexpansive,

iii) for fixed \( x \in H \), \( T(x, \cdot) \) is compact.

Recall that \( f: H \to H \) is nonexpansive if \( ||fx - fy|| \leq ||x - y|| \)
for all \( x, y \in H \).

The following is a special case of a well-known theorem of Browder [8]. (We state it in Hilbert space).

Let \( C \) be a closed, bounded, convex subset of a Hilbert space \( H \) and let \( f: C \to C \) be a semicontractive mapping. Then \( f \) has a fixed point.

The following more general result holds.

**THEOREM 1.** Let \( C \) be a nonempty, closed, convex subset of a Hilbert space \( H \) and let \( f: C \to H \) be semicontractive mapping such that \( f(C) \) is bounded. Then there exists a \( y \in C \) such that

\[
\|y - fy\| = d(fy, C).
\]

**PROOF:** Let \( P: H \to C \) be the proximity map. Then \( P \) is a nonexpansive map, i.e.

\[
\|Px - Py\| \leq \|x - y\| \text{ for all } x, y \in H. \quad (\text{see [9]})
\]

Also,

\[
Pof: C \to C.
\]

Let \( B = \overline{C_0(f(C))} \), convex closure of \( (Pf(C)) \).

Then \( Pf: B \to B \) is a semicontractive mapping and has a fixed point say \( Pfy = y \).

Therefore \( \|y - fy\| = \|Pfy - fy\|

\[
= d(fy, C).
\]

**COROLLARY 1.**

Let \( C \) be a closed, bounded and convex subset of \( H \) and let \( f: C \to H \) be a semicontractive. Then there exists a \( y \in C \) such that

\[
\|y - fy\| = d(fy, C).
\]

Let us now recall the "inwardness condition". Let \( K \) be a closed subset of a Banach space \( X \). We say that \( f: K \to X \) is an inward mapping if for every \( x \in K \)
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\[ f(x) \in I_K(x) = \{ z : z = x + \alpha(y - x), \alpha \geq 0 \} \]

This condition introduced by Halpern [10] and [11] is weaker than \( x \in \delta K \Rightarrow f(x) \in K \) and is widely used in order to obtain fixed point results for mappings \( f : K \to X \). See e.g. Assad and Kirk [12], Caristi [13], Caristi and Kirk [14], Downing and Kirk [15], S. Reich [3], Downing and Ray [16] and S. Massa [17], [18]. (\( \delta K \) stands for boundary of \( K \)).

S. Massa [18] pointed out that if \( K \) is a convex set \( C (K=C) \) then the inwardness condition is equivalent to

\[ x \in C \Rightarrow (x, fy) \cap C \neq \emptyset \]

where \( (x, y) = [(1 - \alpha)x + \alpha y, 0 < \alpha \leq 1] \).

THEOREM 2. Let \( C \) be a closed, convex subset of a Hilbert space \( H \) and \( f : C \to H \) be a semicontractive inward mapping with bounded range. Then \( f \) has a fixed point.

PROOF. Let \( y \in C \) be such that

\[ \|y - fy\| = d(fy, C) \quad \text{(By Theorem 1)}. \]

Suppose \( y \neq fy \). Then \( fy \notin C \) and there exists a \( z \in (y, fy) \cap C \). We have

\[ \|y - fy\| = \|y - z\| + \|z - fy\|. \]

Then \( d(fy, C) \geq \|y - z\| + d(fy, C) \)

absurd, because \( y \neq z \).

COROLLARY 2.

Let \( C \) be a closed, convex subset of \( H \) and let \( f : C \to H \) be semicontractive with bounded range. If \( f(\delta C) \subseteq C \), then \( f \) has a fixed point.

COROLLARY 3.

Let \( B_r \) be a closed ball of radius \( r \) and center \( 0 \) in a Hilbert space \( H \). Let \( f : B_r \to H \) be a semicontractive mapping satisfying the condition: if \( fx = \alpha x \) for \( x \in \delta B_r \), then \( \alpha \leq 1 \). Then \( f \) has a fixed point.

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