
KEY WORDS AND PHRASES. Smooth structures, differential classification, internal groups.


1. INTRODUCTION

Let $E$ represent p-sphere bundle over a q-sphere with $\beta \in \pi_{q-1}SO(p+1)$ the characteristic class of the corresponding p+1-disc bundle over the q-sphere. In [4] R. De Sapio gave a complete classification of the special case where $\beta = 0$. In [5] and [6] Kawakubo and Schultz respectively also gave a classification of $E$ for this special case. This author in [7] gave a generalization of this special case to product of three ordinary spheres. In [1] a classification of $E$ was given for $p < q - 1$ and where $E$ has a cross-section and $\beta \neq 0$. In [3] Schultz gave a classification of $E$ for $p \geq q$ and $E$ is without cross-section. We shall here remove the fact that $E$ has a cross-section so that not every element of $\pi_{q-1}SO(p+1)$ can be pulled back to the element $\pi_{q-1}SO(p)$ in the homomorphism $s_* : \pi_{q-1}SO(p) \to \pi_{q-1}SO(p+1)$ induced by the inclusion $s : SO(p) \to SO(p+1)$. $S^n$ denotes the unit n-sphere with the usual differential structure in the Euclidean
(n+1)-space $R^{n+1} \times \mathbb{Z}^n$ denotes an homotopy n-sphere and $\theta^n$ denotes the group of homotopy n-spheres. $H(p,k)$ denotes the subset of $\theta^p$ which consists of those homotopy p-sphere $\Sigma^p$ such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [4, Lemma 4], $H(p,k)$ is a subgroup of $\theta^p$ and it is not always zero and in fact in [7] we showed that if $k > p-3$, $H(p,k) = \theta^p$. We shall adopt the notation $E(\mathbb{Z}^q)$ to represent the total space of a p-sphere bundle over a homotopy q-sphere $\mathbb{Z}^q$. We will then prove the following:

**THEOREM.** If $M$ is a smooth, n-manifold homeomorphic to a p-sphere bundle over a q-sphere with total space $E$ where $n = p+q \geq 6$ and $p < q$ then there exists homotopy spheres $\mathbb{Z}^q$ and $\mathbb{Z}^n$ such that $M$ is diffeomorphic to $E(\mathbb{Z}^q) \# \mathbb{Z}^n$. We shall define a pairing

$$G : \pi_p SO(q) \times \pi_{p-1} SO(p+1) \to \theta^{p+q}$$

and show that if $\beta \in \pi_{p-1} SO(p+1)$ is the characteristic class of a p-sphere bundle over an homotopy q-sphere $\mathbb{Z}^q$, then $G(\pi_p SO(q), \beta)$ equals the inertial group of $E(\mathbb{Z}^q)$. The above theorem together with the latter will give us the following.

**THEOREM.** Let $E$ be the total space of a p-sphere bundle over a q-sphere then the diffeomorphism classes of $(p+q)$-manifolds that are homeomorphic to $E$ are in one-to-one correspondence with the group

$$H(p,q) \times \text{Image } G_{\beta}$$

where $n = p+q \geq 6$ and $p < q$.

2. **CLASSIFICATION THEOREM**

In this section, we will prove the classification theorem for any manifold $M^n$ homeomorphic to $E$. We will apply the obstruction theory to smoothing of manifolds developed by Munkres in [8]. Since $p+q \geq 6$ and $2 \leq p < q$ then $E$ is simply-connected and the homology of $E$ has no 2-torsion, hence the "Hauptvermutung" of D. Sullivan [9] applies and this means that piecewise linear homeomorphism can be replaced by homeomorphism, we shall not distinguish the two.

**DEFINITION.** Let $M$ and $N$ be smooth closed n-manifolds and $L$ a closed subset of $M$ of dimension less than $n$. Let $f : M \to N$ be a homeomorphism such that for each simplex $\gamma$ of $L$, $f(\gamma)$ are contained in coordinate systems under which they are flat. $f$ is said to be a diffeomorphism modulo $L$ if $f|(M-L)$ is a diffeomorphism and each simplex $\gamma$ of $L$ has a neighborhood $V$ such that $f$ is smooth on $V-L$ near $\gamma$. By [8, Theorem 2.8], if $M$ and $N$ are homeomorphic then there is a diffeomorphism modulo $(n-1)$-skeleton of $M$. If $f : M \to N$ is a diffeomorphism modulo m-skeleton $m < n$ then the obstruction to deforming
f to a diffeomorphism modulo (m-1)-skeleton $g : M \to N$ is an element $\lambda(f) \in H_m(M, r_{n-m})$ where $r_{n-m}$ is a group of diffeomorphism of $S^{n-m-1}$ modulo those that extend to diffeomorphisms of $D^{n-m}$. $g$ is called the smoothing of $f$. If $\lambda(f) = 0$ then by [8, §4] smoothing $g$ exist.

**Theorem 2.1.** If $M$ is a smooth $n$-manifold homeomorphic to $E$ where $E$ denotes the total space of a $p$-sphere bundle over a $q$-sphere, $2 \leq p < q$ and $n = p + q$ then there exist homotopy spheres $\beta^q$ and $\beta^n$ such that $M$ is diffeomorphic to $E(\beta^q) \# \beta^n$ where $E(\beta^q)$ denotes the total space of a $p$-sphere bundle over the homotopy $q$-sphere $\beta^q$.

**Proof.** $E$ is the total space of a $p$-sphere bundle over a $q$-sphere with characteristic class $[b] \in \pi_q SO(p+1)$ then $E = D^q \times S^p \cup D^q \times S^p$ where $f_b : S^{q-1} \times S^p \to S^p \times S^{q-1} \times S^p$ is a diffeomorphism defined by $f_b(x, y) = (x, b(x) \cdot y), (x, y) \in S^{q-1} \times S^p$

$$H_i(E) = \begin{cases} Z & \text{for } i = 0, q, p + q \\ 0 & \text{elsewhere} \end{cases}$$

Since $M^n$ is homeomorphic to $E$ where $n = p + q \geq 6$ and $2 \leq p < q$, then $M^n$ is simply connected and since $H_3(M, \mathbb{Z})$ has no 2-torsion, then "Hauptvermutung" of D. Sullivan [9] implies that there is a piecewise linear homeomorphism $h : M^n \to E$ which by [8, §5] is a diffeomorphism modulo (n-1)-skeleton. Since $H_i(M, \mathbb{Z}) = 0$ for $n - p + 1 \leq i \leq n - 1$ then we can assume that $h$ is a diffeomorphism modulo $n - p = q$ skeleton. The obstruction to a diffeomorphism modulo $q$-1 skeleton is $\lambda(h) \in H_q(M, r^D_p) = r^D_p$. If $[\phi] = \lambda(h) \in r^D_p$ where $\phi : S^{p-1} \to S^{p-1}$ is a diffeomorphism that represents $\lambda(h)$ and let $\beta^p$ denote the homotopy $p$-sphere where $\beta^p = D^p_1 \cup D^p_2$. We define a map

$$j : S^p \to \beta^p \text{ where } S^p = D^p_1 \cup D^p_2$$

such that

$$j(x) = \begin{cases} x & \text{if } x \in D^p_1 \\ \phi^{-1}(\frac{x}{|x|}) & \text{if } x \in D^p_2. \end{cases}$$

So $j$ is an homeomorphism which is identity on $D^p_1$ and the radial extension of $\phi^{-1}$ on $D^p_2$ and so the first obstruction $\lambda(j)$ to deforming $j$ to a diffeomorphism is $[\phi^{-1}] = -\lambda(h)$. We then define $\text{id} \times j : D^q \times S^p \to D^q \times \beta^p$ where $\text{id}$ is the identity, then $\text{id} \times j$ is a homeomorphism and it follows from [8, Def. 3.4] that the first obstruction $\lambda(\text{id} \times j)$ to
deforming \( \text{id} \times j \) to a diffeomorphism is also \(-\lambda(h)\). We can form a manifold \( E' \) by identifying two copies of \( D^q \times S^p \) along their common boundaries \( S^{q-1} \times S^p \) by the diffeomorphism \( f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p \) where \( f_b(x,y) = (x,b(x),y) \) and \([b] \in \pi_{q-1}(SO(p+1)). \) So \( E' = D^q \times S^p \cup D^q \times S^p \). We define a map

\[
g \in (D^q \times S^p)_1 \cup_{f_b} (D^q \times S^p)_2, \quad (D^q \times S^p)_1 \cup_{f_b} (D^q \times S^p)_2 \]

on both \((D^q \times S^p)_1, \) and \((D^q \times S^p)_2, \) the map looks like

\[
\begin{align*}
g & : E' = (D^q \times S^p)_1 \cup_{f_b} (D^q \times S^p)_2 = (D^q \times S^p)_1 \cup_{f_b} (D^q \times S^p)_2 \\
g & : (D^q \times S^p)_1 \cup_{f_b} (D^q \times S^p)_2 \to (D^q \times S^p)_1 \cup_{f_b} (D^q \times S^p)_2
\end{align*}
\]

\( g \) is a homeomorphism and the first obstruction to a diffeomorphism is \( \lambda(\text{id} \times j) = -\lambda(h). \)

It follows that the obstruction to smoothing the composition \( g \cdot h : M \to E' \) is

\[
\lambda(g \cdot h) = \lambda(g) + \lambda(h) = -\lambda(h) + \lambda(h) = 0.
\]

It follows that \( g \cdot h : M \to E' \) is a diffeomorphism modulo \((q-1)\)-skeleton. However in [7, Remark 1] we showed that \( D^q \times S^p \) is diffeomorphic to \( D^q \times S^p \) if \( p < q + 2 \) and so by our hypothesis \( p < q \) then it follows that \( D^q \times S^p \) is diffeomorphic to \( D^q \times S^p \). This implies that \( E \) and \( E' \) are diffeomorphic hence \( g' : M \to E \) is a diffeomorphism modulo \((q-1)\)-skeleton. Since \( H_i(M,Z) = 0 \) for \( p + 1 < i < q - 1 \), there is no more obstruction to deforming \( g' \) to a diffeomorphism until we get to \((p-1)\) skeleton. We can then assume that \( g' \) is a diffeomorphism modulo \( p \)-skeleton. The first obstruction to deforming \( g' \) to a diffeomorphism modulo \((p-1)\)-skeleton is \( \lambda(g') \in H_p(M \times S^q) = r^p \). Let \( [g] = (g') \in r^q \) where \( \phi : S^{q-1} \to S^{q-1} \) is a diffeomorphism which represents \( \lambda(g') \in r^q \). We define \((\phi \times \text{id}) : S^{q-1} \times S^p \to S^{q-1} \times S^p \) where \((\phi \times \text{id})(x,y) = (\phi(x),y)\) and if \( b \in SO(p+1) \), we also define \( f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p \) where \( f_b(x,y) = (x,b(x),y) \). We then have two orientation preserving diffeomorphisms of \( S^{q-1} \times S^p \) unto itself which we can compose to get \( (\phi \times \text{id}) \cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p \) where \((\phi \times \text{id}) \cdot f_b(x,y) = (\phi(x),b(x),y) \). We then construct a manifold by attaching two copies of \( D^q \times S^p \) along their common boundary \( S^{q-1} \times S^p \) using the diffeomorphism \((\phi \times \text{id}) \cdot f_b \) to have \( D^q \times S^p \cup_{f_b} D^q \times S^p \). Notice that this manifold is a \( p \)-sphere bundle over a homotopy \( q \)-sphere \( \Sigma^q = D^q_1 \cup_0 D^q_2 \) whose characteristic map is
\( \beta = [b] \in \pi_{q-1}SO(p+1) \). We define a map

\[
\begin{align*}
\text{h} : D^q \times S^p &\cup D^q_2 \times S^p + D^q_1 \times S^p &\cup D^q_2 \times S^p \\
\phi_{d} \circ (x \cdot id) \cdot f_{b} &\quad \text{by} \\
\text{h}(x,y) = \begin{cases} 
(x,y) & \text{if } (x,y) \in D^q_1 \times S^p \\
(x, \phi_{-1}(\frac{x}{|x|}), y) & \text{if } (x,y) \in D^q_2 \times S^p
\end{cases}
\end{align*}
\]

Hence \( h \) is identity on \( D^q_1 \times S^p \) and a radial extension of \( \phi^{-1} \) on \( D^q_2 \). It then follows that \( h \) is a homeomorphism with the first obstruction to a diffeomorphism being \([\phi^{-1}] = -\lambda(g')\). Then by [8, 3.8] the first obstruction to deforming the composition \( g' \circ h \mapsto M \cup D^q \times S^p \) into a diffeomorphism is \( \lambda(g) = \lambda(g', h) = \lambda(g') + \lambda(h) = -\lambda(h) + \lambda(h) = 0 \) and hence \( g \) is a diffeomorphism modulo \((p-1)\)-skeleton. Since \( H_i(M, Z) = 0 \) for \( 0 < i < p \) then we can assume that \( g \) is a diffeomorphism modulo one point.

Since \( D^q_1 \times S^p \cup D^q_2 \times S^p \) is a \( p \)-sphere bundle over a homotopy \( q \)-sphere \( \Sigma^q \) with characteristic map \([b] \in \pi_{q-1}SO(p+1)\), we shall denote it by \( E(\Sigma^q) \).

Since \( g \) is a diffeomorphism modulo one point then it is known that there is an homotopy \( n \)-sphere \( \Sigma^n \) such that \( M \) is diffeomorphic to \( E(\Sigma^q) \# \Sigma^n \). Hence the proof.

3. INERTIAL GROUPS

Since by Theorem 2.1, every manifold homeomorphic to \( E \) is diffeomorphic to \( E(\Sigma^q) \# \Sigma^n \) for some homotopy spheres \( \Sigma^q, \Sigma^n \), classification of such manifolds reduces to classification of manifolds of the form \( E(\Sigma^q) \# \Sigma^n \). To complete this classification, we then need to investigate what happens when we vary the homotopy spheres and in particular we need to investigate the Inertial group of \( E(\Sigma^q) \). We will investigate these in this section.

**Lemma 3.1.** Let \( \Sigma^q_1 \) and \( \Sigma^q_2 \) be homotopy \( q \)-spheres such that \( \Sigma^q_1 = D^q_1 \cup D^q_2 \) \( i = 1, 2 \) then \( E(\Sigma^q_1) \) is diffeomorphic to \( E(\Sigma^q_2) \) if and only if \( \Sigma^q_1 \# \Sigma^q_2 \in H(q,p) \).

**Proof.** Suppose \( E(\Sigma^q_1) \) is diffeomorphic to \( E(\Sigma^q_2) \). This means that \( D^q_1 \times S^p \cup D^q_2 \times S^p \) is diffeomorphic to \( D^q_1 \times S^p \cup D^q_2 \times S^p \) where \( (\phi_{i} \circ \text{id}) \cdot f_{b} \) \( \phi_{i} \circ \text{id} : S^{q-1} \times S^p \to S^{q-1} \times S^p \) is the diffeomorphism defined by \( \phi_{i}(x,y) = (\phi_{i}(x), y) \) and \( f_{b} : S^{q-1} \times S^p \to S^{q-1} \times S^p \) is defined by \( f_{b}(x,y) = (x, b(x)y) \) where \( [b] = \beta \in \pi_{q-1}SO(p+1) \) is the characteristic map of the bundle. The manifold \( E(\Sigma^q_2) \) can be regarded as the boundary of the \((p+1)\)-disc bundle over \( \Sigma^q_2 \) which is denoted by
\[ D^q_1 \times D^{p+1} \bigcup (\phi_2 \times \text{id}) \cdot f_b \cdot D^q_2 \times D^{p+1} = D(S^q). \] So if \( E(S^q) = \text{diffeomorphic to } E(S^q_1) \) then since \( S^q_1 \) can be embedded in \( E(S^q) \) it follows that \( S^q_1 \) embeds in \( E(S^q_2) \). But \( S^q_2 \) naturally embeds in \( E(S^q_2) \) and so we have \( S^q_1 \) and \( S^q_2 \) sitting in \( E(S^q_2) \), if we translate \( S^q_1 \) away from \( S^q_2 \) we can run a tube between them to obtain an embedding \( S^q_1 \oplus (-S^q_2) \) \( \rightarrow E(S^q_2) \) so that the embedding is homotopically trivial and so by the engulfing result of [10, chapter 7] it means that \( S^q_1 \oplus (-S^q_2) \) can be embedded in the interior of a \((p+q+1)\)-disc in \( E(S^q_2) \) and by [11, 3.5] the embedding is isotopic to a nuclear embedding into the interior of \( S^q \times D^{p+1} \). However the embedding \( S^q_1 \oplus (-S^q_2) \) \( \rightarrow S^q \times D^{p+1} \) is an homotopy equivalence, it then follows by Smale’s theorem [12, Theorem 4.1] that \( S^q_1 \oplus (-S^q_2) \) \( \times D^{p+1} \) is diffeomorphic to \( S^q \times D^{p+1} \) and so it follows that \( S^q_1 \oplus (-S^q_2) \times D^p \) is diffeomorphic to \( S^q \times D^p \) hence \( \Sigma^q_1 \oplus (-\Sigma^q_2) \) \( \in H(q,p) \). Conversely suppose \( \Sigma^q_1 \oplus (-\Sigma^q_2) \) \( \in H(q,p) \) then this implies \((\Sigma^q_1 \oplus (-\Sigma^q_2)) \times D^p \) is diffeomorphic to \( S^q \times D^p \). Since \( S^q \times D^p \) embeds in \( R^{p+q+1} \) with trivial normal bundle then it follows that \( \Sigma^q_1 \oplus (-\Sigma^q_2) \) embeds in \( R^{p+q+1} \) with trivial normal bundle. This shows that each \( \Sigma^q_i \) for \( i = 1, 2 \) embeds in \( R^{p+q+1} \) with trivial normal bundle and by [11, 3.5] the embedding is isotopic to an embedding of \( \Sigma^q_i \) into the interior of \( S^q \times D^{p+1} \). However for \( i = 1, 2 \) the embedding \( \Sigma^q_i \oplus S^q_1 \times D^{p+1} \) is an homotopy equivalence hence it follows from [12, Theorem 4.1] that \( \Sigma^q_i \times D^{p+1} \) is diffeomorphic to \( S^q \times D^{p+1} \) which implies that \( \Sigma^q_1 \times D^{p+1} \) is diffeomorphic to \( \Sigma^q_2 \times D^{p+1} \). Now since \( \Sigma^q_i = D^q_1 \cup D^q_2 \) where \( \phi_i : S^q_1 \times D^{p+1} \) represents \( \Sigma^q_i \in \pi_q \) \( i = 1, 2 \), then we can write \( \Sigma^q_i \times D^{p+1} = D^q_1 \times D^{p+1} \bigcup \phi_i \times \text{id} D^q_2 \times D^{p+1} \) where we identify two copies of \( D^q_1 \times D^{p+1} \) along \( S^q_1 \times D^{p+1} \) by the diffeomorphism \( \phi_i \times \text{id} : S^q_1 \times D^{p+1} \rightarrow S^q_1 \times D^{p+1} \) defined by \((\phi_i \times \text{id})(x,y) = (\phi_i(x),y) \) where \((x,y) \in S^q_1 \times D^{p+1} \). So \( \Sigma^q_1 \times D^{p+1} \) is diffeomorphic to \( \Sigma^q_2 \times D^{p+1} \) implies \( D^q_1 \times D^{p+1} \bigcup \phi_1 \times \text{id} D^q_2 \times D^{p+1} \) is diffeomorphic to \( D^q_1 \times D^{p+1} \bigcup \phi_2 \times \text{id} D^q_2 \times D^{p+1} \).

Now consider the manifold \( D(S^q) = D^q_1 \times D^{p+1} \bigcup D^q_2 \times D^{p+1} \) which is a \((p+1)\)-disc bundle over a \( q \)-sphere with characteristic map \( [b] \in \pi_{q-1} SO(p+1) \). We then form the quotient space

\[ D(S^q) \cup \Sigma^q_1 \times D^{p+1} = (D^q_1 \times D^{p+1} \bigcup D^q_2 \times D^{p+1}) \bigcup (D^q_1 \times D^{p+1} \bigcup D^q_2 \times D^{p+1}) \]

by identifying \( D^q_1 \times D^{p+1} \bigcup D^q_2 \times D^{p+1} \bigcup D^q_1 \times D^{p+1} \bigcup D^q_2 \times D^{p+1} \) by the relation \((x,y) = (x,y)(y \in D^q_1 \times D^{p+1}, \ y \in D^q_2 \times D^{p+1}) \). The manifold \( D(S^q) \cup \Sigma^q_2 \times D^{p+1} \) is similarly constructed. Since \( \Sigma^q_1 \times D^{p+1} \) is diffeomorphic to \( \Sigma^q_2 \times D^{p+1} \). Let \( d : \Sigma^q_1 \times D^{p+1} + \Sigma^q_2 \times D^{p+1} \) be the
diffeomorphism and since any diffeomorphism fixes a disc, we can assume that \( d \) is identity on the disc \( D^{p+q+1} = D^q \times D^{p+1} \), then we can define a diffeomorphism.

\[
g : D(S^q) \cup \Sigma_1^q \times D^{p+1} \rightarrow D(S^q) \cup \Sigma_2^q \times D^{p+1}
\]

where

\[
g(x) = \begin{cases} 
  d(x) & \text{for } x \in \Sigma_1^q \times D^{p+1} \\
  x & \text{for } x \in D(S^q).
\end{cases}
\]

This means that \( g = d \) on \( \Sigma_1^q \times D^{p+1} \) and identity on \( D(S^q) \). \( g \) is well defined because \( d \) is identity on the disc connecting \( D(S^q) \) and \( \Sigma_1^q \times D^{p+1} \) and \( g \) is a diffeomorphism. The manifold \( D(S^q) \cup \Sigma_1^q \times D^{p+1} \) can be clearly seen as follows. Let \( (\phi_i \times \text{id}) \cdot f_b : S^{q-1} \times D^{p+1} + S^{q-1} \times D^{p+1} \) be the diffeomorphism defined by \( ((\phi_i \times \text{id}) \cdot f_b) (x, y) = (\phi_i(x), b(x) \cdot y) \), \((x, y) \in S^{q-1} \times D^{p+1} \) then attaching two manifolds \( D_+^q \times D^{p+1} \) and \( D_+^q \times D^{p+1} \) by the diffeomorphism \( (\phi_i \times \text{id}) \cdot f_b \) we have \( D_+^q \times D^{p+1} \cup D_+^q \times D^{p+1} \) we get a \((p+1)\)-disc bundle over the homotopy q-sphere \( \Sigma^q_1 = D^q_1 \cup D^q_2 \) \( i = 1, 2 \). However, from the way \( (\phi_i \times \text{id}) \cdot f_b \) is constructed it is easily seen that \( D(S^q) \cup \Sigma_1^q \times D^{p+1} = D_+^q \times D^{p+1} \cup D_+^q \times D^{p+1} = D(S^q) \) hence \( g \) is the diffeomorphism of \( D(S_1^q) \) onto \( D(S_2^q) \) then it follows that \( \partial(D(S^q_1)) = E(\Sigma^q_1) \) is diffeomorphic to \( \partial(D(S^q_2)) = E(\Sigma^q_2) \).

Hence the theorem is proved.

REMARK 1. This theorem implies that \( E(\Sigma^q_1) \) is diffeomorphic to \( E(\Sigma^q_2) \) if and only if \( \Sigma^q_1 \) and \( \Sigma^q_2 \) are equivalent in the quotient group \( \mathfrak{e}^q \mathfrak{h}(q, p) \).

To complete this classification, we need to determine the inertial group of \( E(\Sigma^q) \). The inertial group \( \mathfrak{P}(\mathsf{M}) \) of an oriented closed smooth \( n \)-dimensional manifold \( \mathsf{M} \) is defined to be the subgroup of \( \mathfrak{e}^n \) consisting of those homotopy \( n \)-spheres \( \Sigma^n \) such that \( \mathsf{M} \# \Sigma^n \) diffeomorphic to \( \mathsf{M} \).

Let \( E_\beta \) represent the total space of a \( p \)-sphere bundle over a real q-sphere with characteristic class \( \beta \in \pi_{q-1} SO(p+1) \). In [13] we defined a map \( G_\beta : \pi_p SO(q) \rightarrow \mathfrak{e}^{p+q} \) and showed that the image of this map equals the inertial group of \( E_\beta \) where \( p < q \) and \( E_\beta \) has no cross-section. We shall similarly define a map \( G_{\phi \cdot \beta} : \pi_p SO(q) \rightarrow \mathfrak{e}^{p+q} \) and show that the image of this map equals the inertial group of \( E(\Sigma^q) \) where \( E(\Sigma^q) \) is the total space of \( p \)-sphere bundle over a homotopy sphere \( \Sigma^q_1 = D_1^q \cup D_2^q \). Let \( \alpha \in \pi_p SO(q) \) we define

\[
G_{\phi \cdot \beta}(a) = S^{q-1} \times D^{p+1} \cup_{\phi^{-1} (\phi \times \text{id}) \cdot f_b} D^q \times S^p \text{ where } [a] = \alpha \text{ and } [b] = \beta \in \pi_{q-1} SO(p+1) \text{ and}
\]
$f_{a-1}(\phi \times id) \cdot f_b : S^{q-1} \times S^p + S^{q-1} \times S^p$ is a diffeomorphism defined by $f_{a-1}(\phi \times id) \cdot f_b(x,y) = (a^{-1}(b(x) \cdot y) \cdot (x), b(x) \cdot y)$. One can easily show that $G_{\phi \cdot B}$ is well-defined and that its image is an homotopy $(p+q)$-sphere as similarly shown in [13].

**LEMMA 3.2.** Let $E(\varepsilon^q)$ denote the total space of a $p$-sphere bundle over an homotopy $q$-sphere $\varepsilon^q = D_1^q = D_1^q \cup D_2^q$ with characteristic class $\beta \in \pi_{q-1}SO(p+1)$ then $G_{\phi \cdot \beta}^p(SO(q)) \in E(\varepsilon^q)$.

**PROOF.** If $\varepsilon^{p+q} \in I(E(\varepsilon^q))$ then this means there is a diffeomorphism $d : E(\varepsilon^q) \# \varepsilon^{p+q} \rightarrow E(\varepsilon^q)$, that is,

$$d : (D_1^q \times S^p \cup D_2^{q+x} \times S^p) \# \varepsilon^{p+q} \rightarrow D_1^q \times S^p \cup (\phi \times id) \cdot f_b D_2^q \times S^p$$

since $p < q$ then $\pi_p(E(\varepsilon^q))$ is infinitely cyclic and $d(a \times S^q)$ represents a generator and so is homotopic to the inclusion $0 \times S^p + E(\varepsilon^q)$. By Haefliger's theorem [14], $d|0 \times S^p$ and the inclusion $0 \times S^p + E(\varepsilon^q)$ are isotopic and by isotopy extension theorem and tubular neighborhood theorem, $d$ is isotopic to a map which we shall again denote by $d$ such that $d|D^q \times S^p = D^q \times S^p$ where $d(x,y) = (a(y) \times x, y)$ for $[a] \in \pi_p SO(q)$ and $(x,y) \in D^q \times S^p$. We now remove $D^q \times S^p$ from $E(\varepsilon^q) \# \varepsilon^{p+q} = (D_1^q \times S^p \cup D_2^{q+x} \times S^p) \# \varepsilon^{p+q}$ by $(\phi \times id) \cdot f_b$ surgery away from the connected sum and replace it with $S^{q-1} \times D^{p+1}$. After this operation on the summand $E(\varepsilon^q)$ of the connected sum, we have the manifold $S^{q-1} \times D^{p+1}$

$$\cup (\phi \times id) \cdot f_b D^q \times S^p.$$

Since the diffeomorphism $(\phi \times id) \cdot f_b : S^{q-1} \times S^p + S^{q-1} \times S^p$ extend to the diffeomorphism of $S^{q-1} \times D^{p+1}$ onto itself then $S^{q-1} \times D^{p+1} \cup D^q \times S^p$ is diffeomorphic to $S^{q-1} \times D^{p+1} \cup D^q \times S^p$, the diffeomorphism $g$ is defined thus

$$g(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in D^q \times S^p \\ ((\phi \times id) \cdot f_b)(x,y) & \text{if } (x,y) \in S^{q-1} \times D^{p+1}. \end{cases}$$

However, by [7, Lemma 2.1.2], $S^{q-1} \times D^{p+1} \cup D^q \times S^p$ is diffeomorphic to the standard $(p+q)$-sphere $\varepsilon^{p+q}$, hence after this surgery $E(\varepsilon^q)$ is reduced to $\varepsilon^{p+q}$ and so $E(\varepsilon^q) \# \varepsilon^{p+q}$ is reduced to $\varepsilon^{p+q} \times \varepsilon^{p+q} = \varepsilon^{p+q} = \varepsilon^{p+q}$. 


We perform the corresponding modification (under d) on E(\(\mathcal{E}^q\)) to remove the p-sphere 0 x \(S^p\) with product structure d(\(D_q^1\times\mathbb{S}^p\)) in E(\(\mathcal{E}^q\)). From this modification we obtain a manifold \(S^q-1\times D^{p+1}\cup D^q\times S^p\) where \(\psi = (d^{-1}|_{S^q-1\times S^p}).(\phi\times id).f_b\) and this is diffeomorphic to \(\Sigma^{p+q}\) because of the way we performed the surgery using d. However, this manifold \(S^q-1\times D^{p+1}\cup D^q\times S^p = G_{\phi^*\beta}(\mathfrak{a})\) by the definition of \(G_{\phi^*\beta}\), thus there exists an element \(\alpha \in \pi_p SO(q)\) (namely) d|\((D_q^1\times \mathbb{S}^p)\) which gives \(\alpha \in \pi_p SO(q)\) such that \(\Sigma^{p+q} = G_{\phi^*\beta}(\mathfrak{a})\) and so \(\Sigma^{p+q} \in G_{\phi^*\beta}(\pi_p SO(q))\), hence I(E(\(\mathcal{E}^q\))) \subseteq G_{\phi^*\beta}(\pi_p SO(q))\). Conversely suppose \(\Sigma^{p+q} \in G_{\phi^*\beta}(\pi_p SO(q))\) then for some \(\alpha \in \pi_p SO(q)\), \(\Sigma^{p+q} = S^q-1\times D^{p+1}\cup f_{a^{-1}}\mathfrak{a}\times id\cdot f_b\)

\(D^q\times S^p\) where \(\phi\) is a diffeomorphism of \(S^q-1\) onto itself representing \(\mathcal{E}^q = D^q_1\cup D^q_2\) and \(f_{a^{-1}}\) and \(f_b\) are as defined earlier. Notice that \(G_{\phi^*\beta}(\mathfrak{a})\) is thus the obstruction to the construction of a diffeomorphism \(\Sigma^{p+q} \rightarrow \Sigma^{p+q}\). To construct a diffeomorphism from \(\Sigma^{p+q} \rightarrow \Sigma^{p+q}\), we map \(S^q-1\times D^{p+1} \subseteq S^{p+q}\) to itself using \((\phi\times id).f_b\) to have...

and try to extend it to \(D^q\times S^p\). On the boundary \(S^q-1\times S^p\) of \(D^q\times S^p\), the map is \(f_{b^{-1}}\mathfrak{a}\times id\cdot f_a\times id\cdot f_b\). So this means that \(\Sigma^{p+q} = G_{\phi^*\beta}(\mathfrak{a})\) is the obstruction to extending the diffeomorphism \(f_{b^{-1}}\mathfrak{a}\times id\cdot f_a\times id\cdot f_b : S^q-1\times S^p \rightarrow S^{q-1}\times S^p\) to a diffeomorphism of \(D^q\times S^p\) onto itself. We can then define a map \(E(\mathcal{E}^p) \rightarrow E(\mathcal{E}^q)\) using the diffeomorphism \(f_a : D^q_1\times S^p \rightarrow D^q_1\times S^p\) where \(f_a(x,y) = (a(y),x,y)\) and \(D^q_1\times S^p\) we then have...

\[E(\mathcal{E}^q) = D^q_1\times S^p \cup D_2^q\times S^p\]

\[\downarrow f_a\]

\[E(\mathcal{E}^q) = D^q_1\times S^p \cup D_2^q\times S^p\]

On the boundary \(S^q-1\times S^p\) of \(D^q_1\times S^p\), this map is \(f_{b^{-1}}\mathfrak{a}\times id\cdot f_a\times id\cdot f_b\) and the obstruction to extending this to a diffeomorphism of \(E(\mathcal{E}^q)\) onto itself is the...
obstruction to extending the map \( f^{-1} \cdot (\phi^{-1} \times \text{id}) \cdot f \) to the diffeomorphism of
\( D^q \times S^p \) onto itself which is \( \mathbb{Z}^{D+q} \). It then follows that \( E(\mathbb{Z}^q) \to E(\mathbb{Z}^q) \neq \mathbb{Z}^{D+q} \) is a
diffeomorphism and so \( \mathbb{Z}^{D+q} \in I(E(\mathbb{Z}^q)) \) hence
\[
E(E(\mathbb{Z}^q)) = \mathbb{Z}^{D+q} \leq G_q \beta \cdot \mathbb{Z}^p \cdot SO(q)
\]

REMARK 2. We note that if \( p = 2, 4, 5, 6 \) (mod 8) and \( p < q-1 \) then \( \pi_q SO(q) = 0 \) and
so the image of \( G \) is trivial and hence in this particular case, the inertial group of
\( E(\mathbb{Z}^q) \) is trivial and this coincides with the result of [4, Proposition 1].

REMARK 3. By [15], inertial group \( I(M) \) of a smooth manifold \( M \) is a diffeotopy
invariant of \( M \). So if \( 2p \geq q+1 \) then we can deduce that the inertial group \( I(E(\mathbb{Z}^q)) \) of a
\( p \)-sphere bundle over an homotopy \( q \)-sphere \( \mathbb{Z}^q \) is equal to the inertial group \( I(E_B) \) of a
\( p \)-sphere bundle over the standard \( q \)-sphere, where \( \beta \in \pi_{q-1} SO(p+1) \) classifies the
associated disc bundle. Let \( D(\mathbb{Z}^q) \) be the associated \( (p+1) \)-disc bundle over the homotopy
\( q \)-sphere where \( E(\mathbb{Z}^q) \) is the boundary of \( D(\mathbb{Z}^q) \). \( \mathbb{Z}^q \) has the homotopy type of \( D(\mathbb{Z}^q) \) and \( \mathbb{Z}^q \)
has the homotopy type of \( S^q \), it follows that \( S^q \) has the homotopy type of \( D(\mathbb{Z}^q) \). Since
\( 2p \geq q+1 \) then it follows that \( 2(p+q+1) \geq 3q + 3 \) and since \( p+q > 5 \) and \( p \geq 3 \) then \( D(\mathbb{Z}^q) \)
and \( E(\mathbb{Z}^q) \) are simply connected and from [12: Theorem 4.4], it follows that \( D(\mathbb{Z}^q) \) is
diffeomorphic to a \((p+1)\)-disc bundle \( D(S^q) \) over the \( q \)-sphere \( S^q \) hence the boundary
\( \partial D(\mathbb{Z}^q) = E(\mathbb{Z}^q) \) of \( D(\mathbb{Z}^q) \) is diffeomorphic to the boundary \( \partial D(S^q) = E_B \) of \( D(S^q) \). It then
follows by [15] that \( I(E(\mathbb{Z}^q)) = I(E_B) \). This means that the inertial group of \( S^q \) in [13]
coincides with Lemma 3.2.

Combination of Lemmas 3.1 and 3.2 give the following.

THEOREM 3.3. Let \( E \) be the total space of a \( p \)-sphere bundle over a \( q \)-sphere with
characteristic map \( \beta \in \pi_{q-1} SO(p+1) \) then the diffeomorphism classes of \( p+q \)-manifolds that
are homeomorphic to \( E \) are in one-to-one correspondence with the group
\[
\mathbb{Z}^q \times \mathbb{Z}^n \cong \mathcal{H}(q,p) \times \text{Image} \ \mathcal{G}_B
\]
where \( p+q = n \geq 6 \) and \( p < q \).

REFERENCES
