CONVERGENCE OF THE SOLUTIONS FOR THE EQUATION

\[
x^{(iv)} + a\dddot{x} + b\dot{x} + g(x) + h(x) = p(t,x,\dot{x},\ddot{x})
\]

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(Received May 30, 1984 and in revised form November 26, 1985)

ABSTRACT. This paper is concerned with differential equations of the form

\[
x^{(iv)} + a\dddot{x} + b\dot{x} + g(x) + h(x) = p(t,x,\dot{x},\ddot{x})
\]

where a, b are positive constants and the functions g, h and p are continuous in their respective arguments, with the function h not necessarily differentiable. By introducing a Lyapunov function, as well as restricting the incrementary ratio \( \frac{h(\xi + \eta) - h(\xi)}{\eta} \) to a closed sub-interval of the Routh-Hurwitz interval, we prove the convergence of solutions for this equation. This generalizes earlier results.

KEY WORDS AND PHRASES. Routh-Hurwitz interval, Lyapunov function.

1980 AMS SUBJECT CLASSIFICATION CODE. 34D20.

1. INTRODUCTION.

Consider fourth-order differential equations of the form:

\[
x^{(iv)} + a\dddot{x} + b\dot{x} + g(x) + h(x) = p(t,x,\dot{x},\ddot{x})
\]

in which a > 0, b > 0, functions g and h are continuous in their respective arguments. The function \( p(t,x,\dot{x},\dddot{x}) \) is assumed to have the form \( q(t) + r(t,x,\dot{x},\dddot{x}) \) with the functions q and r depending explicitly on the arguments displayed, and continuous in their respective arguments. Further, we shall assume that \( r(t,0,0,0,0) = 0 \) for all t.

The solutions of (1.1) will be said to converge if any two solutions \( x_1(t), x_2(t) \) of (1.1) satisfy

\[
\begin{align*}
x_2(t) - x_1(t) & \to 0, \quad \dot{x}_2(t) - \dot{x}_1(t) \to 0, \\
\dddot{x}_2(t) - \dddot{x}_1(t) & \to 0, \quad \ddot{x}_2(t) - \ddot{x}_1(t) \to 0,
\end{align*}
\]

as \( t \to \infty \).

The convergence of solutions for equations of the form (1.1) was earlier shown in [1], when \( g(x) = cx \), with \( c > 0 \), and with the assumption that \( h(x) \) is not necessarily differentiable, but with an incrementary ratio \( \frac{h(\xi + \eta) - h(\xi)}{\eta} \),
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\( \eta \neq 0 \), lying in a closed sub-interval \( I_0 \) of the Routh-Hurwitz interval \( (0, (ab - c)c/a^2) \), where

\[
I_0 = \left[ \frac{A_0}{2}, \frac{K(ab - c)c}{a^2} \right]
\]  

(1.3)

\( A_0 > 0 \) and \( K < 1.0 \).

The main purpose of the present investigation is to give fourth-order analogues of [2], as well as extending earlier results in [1] to equations of the form (1.1) with the additional condition that for \( y_1 \neq y_2 \),

\[
c_0 > \frac{g(y_2) - g(y_1)}{y_2 - y_1} > c
\]

(1.4)

for some constants \( c_0 > 0 \) and \( c > 0 \), satisfying

\[
abc > c_0^2
\]

(1.5)

Moreover, while proving the convergence results for (1.1), we shall give a general estimate for the constant \( K < 1 \), from which a particular case is derived.

2. MAIN RESULTS.

The main results of this paper, which are in some respects fourth-order analogues of [2] and generalizations of [1], are the following:

THEOREM 1. Suppose that \( g(0) = h(0) \) and that

(i) there are constants \( c > 0, c_0 > 0 \) such that \( g(y) \) satisfies inequalities (1.4) and (1.5);

(ii) there are constants \( A_0 > 0, K < 1 \) such that for any \( \xi, \eta, (\eta \neq 0) \), the incrementary ratio for \( h \) satisfies

\[
\eta^{-1}(h(\xi + \eta) - h(\xi)) \text{ lies in } I_0
\]

(2.1)

with \( I_0 \) as defined in (1.3);

(iii) there is a continuous function \( \phi(t) \) such that

\[
|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|
\]

(2.2)

holds for arbitrary \( t, x_1, y_1, z_1, w_1, x_2, y_2, z_2, \) and \( w_2 \).

Then, there exists a constant \( D_1 \) such that if

\[
\int_0^t \phi^\alpha(\tau)d\tau \leq D_1 t
\]

(2.3)

for some \( \alpha \), in the range \( 1 < \alpha < 2 \), then all solutions of (1.1) converge.

A very important step in the proof of Theorem 1 will be to give estimates for any two solutions of (1.1). This in itself, being of independent interest, is given as:
THEOREM 2. Let $x_1(t)$, $x_2(t)$ be any two solutions of (1.1). Suppose that all the conditions of Theorem 1 are satisfied, then for each fixed $\alpha$, in the range $1 < \alpha < 2$, there exist constants $D_2$, $D_3$ and $D_4$ such that for $t_2 > t_1$, 

$$S(t_2) \leq D_2 S(t_1) \exp \left\{ -D_3 (t_2 - t_1) + \frac{t_2}{t_1} \phi^\alpha(\tau) d\tau \right\}$$

where

$$S(t) = \left\{ \left[ x_2(t) - x_1(t) \right]^2 + \left[ \dot{x}_2(t) - \dot{x}_1(t) \right]^2 + \left[ \ddot{x}_2(t) - \ddot{x}_1(t) \right]^2 + \left[ \dddot{x}_2(t) - \dddot{x}_1(t) \right]^2 \right\}.$$  

(2.5)

If we put $x_1(t) = 0$ and $t_1 = 0$, we immediately obtain:

COROLLARY 1. If $\alpha = 0$ and the hypotheses (i) and (ii) of Theorem 1 hold, then the trivial solution of (1.1) is exponentially stable in the large.

Further, if we put $\xi = 0$ in (2.1) with $n (n \neq 0)$ arbitrary, we obtain:

COROLLARY 2. If $\alpha = 0$ and the hypotheses (i) and (ii) hold for arbitrary $n (n \neq 0)$, and $\xi = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of (1.1) satisfies

$$|x(t)| \leq D_5; |\dot{x}(t)| \leq D_5; |\ddot{x}(t)| \leq D_5; |\dddot{x}(t)| \leq D_5.$$  

(2.6)

3. PRELIMINARY RESULTS.

Let $Q(t) = \int_0^t q(\tau) d\tau$. For convenience, by setting $\dot{x} = y$, $\dot{y} = z$ and $\dot{z} = w + Q(t)$, we replace equation (1.1) by the equivalent system:

$$\begin{align*}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= w + Q(t) \\
\dot{w} &= -aw - bz - g(y) - h(x) - r(t, x, y, z, w+Q(t)) - aQ(t)
\end{align*}$$

(3.1)

Let $(x_i(t), y_i(t), z_i(t), w_i(t)), (i=1,2)$, be two solutions of (3.1), such that

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0;$$

(3.2)

and

$$\Delta_0 \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq \frac{K(ab - c)c}{a^2}$$

(3.3)

where $c$, $c_0$, $\Delta_0$, $K$ are as defined in (1.3), (1.4) and (1.5).

Our main tool in the proofs of the convergence Theorems will be the following function: $W = W(x_2 - x_1, y_2 - y_1, x_2 - z_1, w_2 - w_1)$ defined by

$$2W = \left\{ c^2 e^{(1 - \varepsilon)} (x_2 - x_1)^2 + ac(1 - \varepsilon)(D - 1)(y_2 - y_1)^2 + 2c^2 e^{(D - 1)}(y_2 - y_1)(z_2 - z_1) + eD(w_2 - w_1)^2 + b(D - 1)(z_2 - z_1)^2 + (1 - \varepsilon)D - 1)a(z_2 - z_1) + (w_2 - w_1)^2 + [c(1 - \varepsilon)(x_2 - x_1) + b(y_2 - y_1) + (z_2 - z_1) + (w_2 - w_1)]^2 \right\}.$$
where $D = 1 = (q + cc)/(ab - c - d)$, with $ab - c > 0$, $0 < c < 1$; and $abc(2 - c) = d$. This is an adaptation of the function $V$ used in [1].

Since $0 < c < 1$, following the argument used in [1], we can easily verify the following for $W$.

**LEMMA 1.** (i) $W(0,0,0,0) = 0$; and
(ii) there exist finite constants $D_6 > 0$, $D_7 > 0$ such that

$$D_6 \left\{ (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (w_2-w_1)^2 \right\} < W <$$

$$D_7 \left\{ (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (w_2-w_1)^2 \right\}. \tag{3.5}$$

If we define the function $W(t)$ by $W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t), w_2(t) - w_1(t))$, and using the fact that the solutions $(x_i, y_i, z_i, w_i + Q(t)), (i = 1, 2)$, satisfy (3.1), then $S(t)$ as defined in (2.5) becomes

$$S(t) = [(x_2(t) - x_1(t))^2 + (y_2(t) - y_1(t))^2 + (z_2(t) - z_1(t))^2 + (w_2(t) - w_1(t))^2] \tag{3.6}$$

We can then prove the following result on the derivative of $W(t)$ with respect to $t$.

**LEMMA 2.** Let the hypotheses (i) and (ii) of Theorem 1 hold. Then, there exist positive finite constants $D_8$ and $D_9$ such that

$$\frac{dw}{dt} \leq -2D_8S + D_9S^{1/2} \tag{3.7}$$

where $\Theta = r(t, x_2, y_2, z_2, w_2 + Q) - r(t, x_1, y_1, z_1, w_1 + Q)$.

**PROOF OF LEMMA 2.** On using (3.1), a direct computation of $\frac{dw}{dt}$ gives after simplification

$$\frac{dw}{dt} = -W_1 + W_2 \tag{3.8}$$

where

$$W_1 = \{c(l-\epsilon)H(x_2,x_1)(x_2-x_1)^2 + bcc(y_2-y_1)^2 + abc(1-\epsilon)D(z_2-z_1)^2 +$$

$$+ aed(w_2-w_1)^2\} + \{G(y_2,y_1) - c\{c(l-\epsilon)(x_2-x_1) +$$

$$+ b(y_2-y_1) + a(1-\epsilon)D(z_2-z_1) + D(w_2-w_1)\}(y_2-y_1) +$$

$$+ H(x_2,x_1)\{b(y_2-y_1) + a(1-\epsilon)D(z_2-z_1) + D(w_2-w_1)\}(x_2-x_1)$$

and

$$W_2 = \Theta(t)\{c(l-\epsilon)(x_2-x_1) + b(y_2-y_1) + a(1-\epsilon)D(z_2-z_1) + D(w_2-w_1)\},$$

with

$$G(y_2,y_1) = \frac{g(y_2) - g(y_1)}{y_2 - y_1}, \quad (y_2 \neq y_1); \tag{3.9}$$

$$H(x_2,x_1) = \frac{h(x_2) - h(x_1)}{x_2 - x_1}, \quad (x_2 \neq x_1). \tag{3.10}$$
Let \( \lambda = (g(y_2,y_1) - c) > 0 \) for \( y_2 \neq y_1 \). Define

\[
\sum_{i=1}^{5} a_i = 1 ; \quad \sum_{i=1}^{5} \beta_i = 1 ; \quad \sum_{j=1}^{3} \gamma_j = 1 \quad \text{and} \quad \sum_{j=1}^{3} \delta_j = 1 ,
\]

with \( a_i > 0, \beta_i > 0, \gamma_j > 0 \) and \( \delta_j > 0 \). Further, let us denote \( \mathcal{H}(x_2,x_1) \) simply by \( \mathcal{H} \). Then, we can re-arrange \( W_1 \) as

\[
W_1 = W_{11} + W_{12} + W_{13} + W_{14} + W_{21} + W_{23} + W_{24}
\]

where

\[
W_{11} = \begin{cases} 
\alpha_1 c(1-\epsilon)\mathcal{H}(x_2-x_1)^2 + b\beta_1 c\epsilon + \lambda(y_2-y_1)^2 \\
+ \gamma_1 ab\epsilon(1-\epsilon)D(z_2-z_1)^2 + \delta_4 a\epsilon D(w_2-w_1)^2
\end{cases}
\]

\[
W_{12} = \begin{cases} 
\beta_2 bce(y_2-y_1)^2 + \alpha_2 c(1-\epsilon)(x_2-x_1)(y_2-y_1) + \\
+ a_2 c(1-\epsilon)\mathcal{H}(x_2-x_1)^2
\end{cases}
\]

\[
W_{13} = \begin{cases} 
\beta_3 bce(y_2-y_1)^2 + \alpha_3 c(1-\epsilon)D(y_2-y_1)(z_2-z_1) + \\
+ \gamma_2 ab\epsilon(1-\epsilon)D(z_2-z_1)^2
\end{cases}
\]

\[
W_{14} = \begin{cases} 
\beta_4 bce(y_2-y_1)^2 + \alpha_4 c(1-\epsilon)\mathcal{H}(x_2-x_1)^2 + b\beta_4 c\epsilon + \lambda(y_2-y_1)^2 \\
+ \gamma_4 ab\epsilon(1-\epsilon)D(z_2-z_1)^2 + \delta_4 a\epsilon D(w_2-w_1)^2
\end{cases}
\]

\[
W_{21} = \begin{cases} 
\alpha_5 c(1-\epsilon)\mathcal{H}(x_2-x_1)^2 + b\beta_5 c\epsilon + \lambda(y_2-y_1)^2 \\
+ a_5 c(1-\epsilon)D(y_2-y_1)(w_2-w_1) + \delta_2 a\epsilon D(w_2-w_1)^2
\end{cases}
\]

\[
W_{23} = \begin{cases} 
\beta_6 bce(y_2-y_1)^2 + \alpha_6 c(1-\epsilon)(x_2-x_1)(z_2-z_1) + \\
+ a_6 c(1-\epsilon)\mathcal{H}(x_2-x_1)(z_2-z_1)
\end{cases}
\]

\[
W_{24} = \begin{cases} 
\beta_7 bce(y_2-y_1)^2 + \alpha_7 c(1-\epsilon)D(y_2-y_1)(w_2-w_1) + \\
+ \gamma_7 ab\epsilon(1-\epsilon)D(w_2-w_1)^2 + \delta_2 a\epsilon D(w_2-w_1)^2
\end{cases}
\]

and

\[
W_{12} = \begin{cases} 
\alpha_3 c(1-\epsilon)\mathcal{H}(x_2-x_1)^2 + b\beta_3 c\epsilon + \lambda(y_2-y_1)^2 \\
+ a_3 c(1-\epsilon)D(y_2-y_1)(z_2-z_1) + \delta_3 a\epsilon D(w_2-w_1)^2
\end{cases}
\]

Each \( W_{ij}, (i \neq j), \) \( (i = 1,2; j = 1,2,3,4) \), is quadratic in its respective variables. Also, using the fact that any quadratic of the form \( Au^2 + Bu + Cv^2 \) is non-negative if \( (4AC - B^2) > 0 \), we obtain that

\[
W_{21} > 0 \quad \text{if} \quad \lambda^2 < \frac{4bc\epsilon\alpha_2 \beta_2}{1 - \epsilon};
\]

\[
W_{23} > 0 \quad \text{if} \quad \lambda^2 < \frac{4b^2 c^2 \beta_3 \gamma_2}{a(1-\epsilon)D};
\]

\[
W_{24} > 0 \quad \text{if} \quad \lambda^2 < \frac{4\alpha_4 c^2 b\gamma_4}{bD};
\]

\[
W_{12} > 0 \quad \text{if} \quad H < \frac{4c^2 \epsilon(1-\epsilon)\alpha_3 \beta_5}{b};
\]

\[
W_{13} > 0 \quad \text{if} \quad H < \frac{4bc\epsilon\alpha_3 \gamma_3}{aD};
\]

and

\[
W_{14} > 0 \quad \text{if} \quad H < \frac{4ace(1-\epsilon)\alpha_5 \delta_3}{D}.
\]
Thus \( W_1 > W_{11} \), provided that
\[
0 < \lambda^2 < 4 \min \left\{ \frac{bc\Delta \alpha_2 \beta_2}{(1 - \varepsilon)} ; \frac{b^2c\beta_2 \gamma_2}{a(1-\varepsilon)D} ; \frac{abc \varepsilon \delta_2 \beta_4}{D} \right\}
\] (3.12)
and
\[
H \text{ lies in } I_2 \equiv \left[ \frac{\Delta_2, K(ab - c)c}{a^2} \right]
\] (3.13)
a closed sub-interval of the Routh-Hurwitz interval \((0, (ab-c)c/a^2)\), with
\[
K = \left( \frac{4}{ab - c} \right) \min \left\{ \frac{ca^2 \varepsilon (1-\varepsilon) \alpha_2 \beta_5}{b} ; \frac{abc \varepsilon \gamma_3}{D} ; \frac{a^3 \varepsilon (1-\varepsilon) \alpha_2 \delta_3}{D} \right\}
\] (3.14)
By choosing \( 2D_8 = \min \{c(1-\varepsilon)\Delta_0; bc\varepsilon; ab\varepsilon (1-\varepsilon)D; a\varepsilon D\} \), we clearly have
\[
W_1 > W_{11} > 2D_8 S
\] (3.15)
also, if we choose \( D_9 = 2 \max \{c(1-\varepsilon); b; a(1-\varepsilon)D; D\} \), we obtain:
\[
W_2 < D_9 S | |0||
\] (3.16)
Combining (3.15) and (3.16) in (3.8), we obtain (3.7). This completes the proof of
Lemma 2.

4. PROOF OF THEOREM 2.
This follows directly from [3], on using inequality (3.7). Let \( \alpha \) be any constant in the range \( 1 < \alpha < 2 \). Set \( 2\mu = 2 - \alpha \), so that \( 0 < 2\mu < 1 \). We re-write (3.7) in the form
\[
\frac{dW}{dt} + D_8 S \leq D_9 S^{2\mu} W^*
\] (4.1)
where \( W^* = (|0| - D_9 D_9^{-1} S^\frac{1}{2}) S^{\frac{1}{2} - \mu} \).

Considering the two cases (i) \( |0| < D_8 S^{1/2} D_9 \) and (ii) \( |0| > D_8 S^{1/2} D_9 \) separately, we find that in either case, there exists some constant \( D_{11} > 0 \) such that
\( W^* < D_{11} | |0|| 2(1-\mu) \). Thus using (2.2), inequality (4.1) becomes
\[
\frac{dW}{dt} + D_8 S \leq D_9 D_9^{1/2} (1-\mu) S(1-\mu),
\] (4.2)
where \( D_{12} > 2D_9 D_{11} \). This immediately gives
\[
\frac{dW}{dt} + (D_{13} - D_9^{\alpha}(t)) W \leq 0
\] (4.3)
after using Lemma 1 on $W$, with $D_{13}$ and $D_{14}$ as some positive constants.

On integrating (4.3) from $t_1$ to $t_2$, $(t_2 > t_1)$, we obtain

$$W(t_2) \leq W(t_1) \exp \{-D_{13}(t_2-t_1) + D_{14} \int_{t_1}^{t_2} \phi^2(t) \, dt\}.$$  \hfill (4.4)

Again, using Lemma 1, we obtain (2.4), with $D_2 = D_7/D_6$, $D_3 = D_{13}$ and $D_4 = D_{14}$.

This completes the proof of Theorem 2.

5. PROOF OF THEOREM 1.

This follows from the estimate (2.4) and the condition (2.3) on $\phi(t)$. Choose $D_1 = D_3/D_4$ in (2.3). Then, as $t = (t_2-t_1) \to \infty$, $S(t) \to 0$, which proves that as $t \to \infty$,

$$x_2(t) - x_1(t) \to 0, \quad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$

$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \quad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0.$$

This completes the proof of Theorem 1.

6. REMARKS.

(i) If in (3.14) we choose

$$\alpha_1 = 1/2; \quad \alpha_j = 1/8 \quad (j = 2, 3, 4, 5);$$

$$\beta_1 = 1/2; \quad \beta_j = 1/8 \quad (j = 2, 3, 4, 5);$$

$$\gamma_1 = 1/2; \quad \gamma_2 = \gamma_3 = 1/4;$$

$$\delta_1 = 1/2; \quad \delta_2 = \delta_3 = 1/4,$$

we obtain

$$K = \left( \frac{1}{16(ab-c)} \right) \min \left\{ \frac{ca^2c(1-e)}{b} \frac{2abc}{D}; \frac{2a^3c(1-e)}{D} \right\} < 1.$$  \hfill (ii)

(ii) As remarked in [1], the results remain valid if we replace $\phi(t)$ in (2.3) by a constant $D_{15} > 0$.

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