S-ASYMPTOTIC EXPANSION OF DISTRIBUTIONS

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ABSTRACT. This paper contains first a definition of the asymptotic expansion at
infinity of distributions belonging to $\mathcal{D}'(\mathbb{R}^n)$, named S-asymptotic expansion, as also
its properties and application to partial differential equations.

KEYS WORDS AND PHRASES. Convex cone, distribution, behaviour of a distribution at
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1. INTRODUCTION.

The basic idea of the asymptotic behaviour at infinity of a distribution one can
find already in the book of L. Schwartz [1]. To these days many mathematicians tried
to find a good definition of the asymptotic behaviour of a distribution. We shall
mention only "equivalence at infinity" explored by Lavoine and Misra [2] and the
"quasiasymptotic" elaborated by Vladimirov and his pupils [3]. Brichkov [4] intro-
duced the asymptotic expansion of tempered distributions as a useful mathematical
tool in quantum field theory. His investigations and definitions were turned just
towards these applications. In [4] one can find cited literature in which asymptotic
expansion technique, introduced by Brichkov, was used in the quantum field theory.
This is a reason to study S-asymptotic expansion.

2. DEFINITION OF THE S-ASYMPTOTIC EXPANSION.

In the classical analysis we say that the sequence $\{\psi_n(t)\}$ of numerical functions
is asymptotic if and only if $\psi_{n+1}(t) = o(\psi_n(t)), t \to \infty$. The formal series $\sum_{n=1}^{\infty} u_n(t)$
is an asymptotic expansion of the function $u(t)$ related to the asymptotic sequence
$\{\psi_n(t)\}$ if

$$u(t) - \sum_{n=1}^{k} u_n(t) = o(\psi_k(t)), t \to \infty$$  \hspace{1cm} (2.1)

for every $k \in \mathbb{N}$ and we write

$$u(t) \sim \sum_{n=1}^{\infty} u_n(t) \mid \{\psi_n(t)\}, t \to \infty$$  \hspace{1cm} (2.2)

When for every $n \in \mathbb{N}$ $u_n(t) = c_n \psi_n(t)$, $c_n$ are complex numbers, expansion (2.2)
is unique, that means the numbers $c_n$ can be determined in only one way.
In this text $\Gamma$ will be a convex cone with vertex at zero belonging to $\mathbb{R}^n$ and $\Sigma(\Gamma)$ the set of all real valued and positive functions $c(h)$, $h \in \Gamma$. Notations for the spaces of distributions are as in the books of Schwartz [1].

**DEFINITION 1.** The distribution $T \in \mathcal{D}'$ has the $S$-asymptotic expansion related to the asymptotic sequence $\{c_n(h)\} \subseteq \Sigma(\Gamma)$, we write it

$$T(t+h) \sim \sum_{n=1}^{\infty} \hat{u}_n(t, h) \mid \{c_n(h)\}, \|h\| \to \infty, h \in \Gamma$$

where $\hat{u}_n(t, h) \in \mathcal{D}'$ for $n \in \mathbb{N}$ and $h \in \Gamma$, if for every $\rho \in \mathcal{D}$

$$\langle T(t+h), \rho(t) - \sum_{n=1}^{\infty} \hat{u}_n(t, h), \rho(t) \rangle = \langle c_n(h), \rho(t) \rangle, \|h\| \to \infty, h \in \Gamma$$

**REMARK.**
1) In the special case $\hat{u}_n(t, h) = u_n(t) c_n(h)$, $u_n \in \mathcal{D}$, $n \in \mathbb{N}$, we shall write

$$T(t+h) \sim \sum_{n=1}^{\infty} u_n(t) c_n(h), \|h\| \to \infty, h \in \Gamma$$

and the given $S$-asymptotic expansion is unique.

2) To define the $S$-asymptotic expansion in $\mathcal{D}'(\mathbb{R}^n)$, we have only to suppose that in relation (2.4) $T$ and $\hat{u}_n$ are in $\mathcal{S}'$ and $\rho$ in $\mathcal{S}$.

Brihkov's general definition is slightly different [5].

**DEFINITION 1'.** The distribution $g \in \mathcal{S}'$ has the asymptotic expansion related to the asymptotic sequence $\{\psi_n(t)\}$ on the ray $\{\lambda h_0, \lambda > 0\}$, $h_0 \in \mathbb{R}^n$

$$g(\lambda h_0 - t) \sim \sum_{n=1}^{\infty} \hat{c}_n(t, \lambda) \mid \{\psi_n(\lambda)\}, \lambda \in \mathbb{R}, \lambda \to \infty$$

where $\hat{c}_n(t, \lambda) \in \mathcal{S}'$ for $\lambda \geq \lambda_0 > 0$, if for every $\phi \in \mathcal{S}$

$$\langle g(\lambda h_0 - t), \phi(t) - \sum_{n=1}^{\infty} \hat{c}_n(t, \lambda), \phi(t) \rangle = \langle \psi_n(\lambda), \phi(t) \rangle, \lambda \to \infty$$

Relation 2.6 can be transformed in

$$f(x) \exp(\lambda x) \sim \sum_{n=1}^{\infty} c_n(x, \lambda) \mid \{\psi_n(\lambda)\}, \lambda \to \infty$$

by the Fourier transform, if we take $f(x) = F^{-1}[g(t)]; \rho(x) = F^{-1}[\phi(t)]$ and $F[\hat{c}_n(t, \lambda)] = (2\pi)^n c_n(x, \lambda)$. We denote by $F[\rho]$ the Fourier transform of $\rho$ and by $F^{-1}[g]$ the inverse Fourier transform of $g$. Also, for $x, t \in \mathbb{R}^n \exp(\lambda x, t) = \sum_{i=1}^{n} x_i t_i$.

In his papers Bričkov considered only the asymptotic expansions (2.8) and in one dimensional case. We shall study the asymptotic expansion not in $\mathcal{S}'(\mathbb{R})$ but in the whole $\mathcal{S}'(\mathbb{R}^n)$, not only on a ray but on a cone in $\mathbb{R}^n$. Our results enlarge Bričkov's to be valued for the elements of $\mathcal{S}'(\mathbb{R}^n)$ (Corollary 1), they are proved with less suppositions (Propositions 5 and 6) or give new properties of the $S$-asymptotic.

A distribution belonging to $\mathcal{S}'(\mathbb{R}^n)$ can have $S$-asymptotic expansion in $\mathcal{S}'$ without having the same $S$-asymptotic expansion in $\mathcal{S}'$. Such an example is the regular distribution $\tilde{f}$ defined by the function

$$f(t) = H(t) \exp(1/(1+t^2)) \exp(-t), \ t \in \mathbb{R}$$
where
\[ H(t) = 1, \quad t \geq 0 \quad \text{and} \quad H(t) = 0, \quad t < 0. \]

It is easy to prove that for \( h \in \mathbb{R}_+ \)
\[ \tilde{f}(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1 + (t+h)^2)^{1-n} \exp(-t-h) \left\{ e^{h^2(1-n)} \right\}, \quad h \to \infty. \]
But
\[ U_n(t,h) = (1 + (t+h)^2)^{1-n} \exp(-t-h), \quad n \in \mathbb{N}, \quad h > 0 \]
do not belong to \( \mathcal{S} \).

The regular distribution \( \tilde{g} \) defined by the function
\[ g(t) = \exp(1 + (1+t^2)) \exp(t), \quad t \in \mathbb{R} \]
belongs to \( \mathcal{D} \) but it is not in \( \mathcal{S} \). It has S-asymptotic expansion in \( \mathcal{D} \):
\[ \tilde{g}(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1 + (t+h)^2)^{1-n} \exp(t+h) \left\{ e^{h^2(1-n)} \right\}, \quad h \to \infty, \quad h \in \mathbb{R}_+ \]
where \( \Gamma = \mathbb{R}_+ \).

3. PROPERTIES OF THE S-ASYMPTOTIC EXPANSION.

**PROPOSITION 1.** Let \( S \in \mathcal{E} \) and \( T \in \mathcal{D} \). If
\[ T(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} U_n(t,h) \left\{ c_n(h) \right\}, \quad \|h\| \to \infty, \quad h \in \Gamma \]
then the convolution
\[ (S * T)(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} (S * U_n)(t,h) \left\{ c_n(h) \right\}, \quad \|h\| \to \infty, \quad h \in \Gamma \]

**PROOF.** We know that
\[ <(S * T)(t+h), \rho(t)> = \sum_{n=1}^{k} <(S * U_n)(t,h), \rho(t)> = <S * [T(t+h) - \sum_{n=1}^{k} U_n(t,h)], \rho(t)> . \]

It remains only to use the continuity of the convolution.

**COROLLARY 1.** If
\[ T(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} U_n(t,h) \left\{ c_n(h) \right\}, \quad \|h\| \to \infty, \quad h \in \Gamma \]
then
\[ T^{(k)}(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} U_n^{(k)}(t,h) \left\{ c_n(h) \right\}, \quad \|h\| \to \infty, \quad h \in \Gamma \]
where \( T^{(k)} = (T_1^{(k)} \ldots T_n^{(k)}) \), \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \), \( N_o = \mathbb{N} \cup \{0\} \).

**PROOF.** We have only to take \( S = \delta^{(k)} \) in Proposition 1.

**REMARK.** Proposition 1. is valued as well if we suppose that \( T \in \mathcal{S} \) and \( S \in \mathcal{D}' \).

**PROPOSITION 2.** Let \( f, U_n(t,h) \) and \( V_n(t) \), \( n \in \mathbb{N} \) and \( h \in \Gamma \), be the local integrable functions such that for every compact set \( K \subset \mathbb{R}^n \)
\[ f(t+h) \overset{S}{\to} \sum_{n=1}^{\infty} U_n(t,h) \left\{ c_n(h) \right\}, \quad \|h\| \to \infty, \quad h \in \Gamma, \quad t \in K \]
and
\[ f(t+h) - \sum_{n=1}^{k} U_n(t,h) | c_k(h) \leq V_k(t), \quad t \in K, \quad h \in \Gamma \]
and \( \|h\| \leq r(k,K) \), then for the regular distribution \( \tilde{f} \) defined by \( f \) we have
\[ \tilde{f}(t+h) \overset{\infty}{\approx} \sum_{n=1}^{\infty} \tilde{U}_n(t,h) | c_n(h), \quad \|h\| \to \infty, \quad h \in \Gamma. \]

**PROOF.** The proof is a consequence of the Lebesgue's theorem.

**PROPOSITION 3.** Suppose that \( T_1 \) and \( T_2 \) belong to \( \mathcal{B} \) and equal over the open set \( \Omega \) which has the property: for every \( r > 0 \) there exists a \( \beta_r \) such that the ball \( B(0,r) = \{ x \in \mathbb{R}^n, \|x\| \leq r \} \) is in \( \{ \Omega, h \in \Gamma, \|h\| \geq \beta_r \} \). If
\[ T_1(t+h) \overset{\infty}{\approx} \sum_{n=1}^{\infty} U_n(t,h) | c_n(h), \quad \|h\| \to \infty, \quad h \in \Gamma, \]
then
\[ T_2(t+h) \overset{\infty}{\approx} \sum_{n=1}^{\infty} U_n(t,h) | c_n(h), \quad \|h\| \to \infty, \quad h \in \Gamma \]
as well.

**PROOF.** We have only to prove that for every \( c_k(h) \)
\[ \lim_{\|h\| \to \infty, h \in \Gamma} \frac{< [T_1(t+h) - T_2(t+h)], c_k(h), \rho(t) >}{c_k(h) \rho(t)} = 0, \quad \rho \in \mathcal{B} \] (3.3)

Let \( \text{supp } \rho \subset B(0,r) \). The distribution \( T_1(t+h) - T_2(t+h) \) equals zero over \( \Omega \). By the supposition there exists a \( \beta_r \) such that the ball \( B(0,r) \) is in \( \{ \Omega, h \in \Gamma, \|h\| \geq \beta_r \} \). This proves out relation (3.3).

**PROPOSITION 4.** Let \( S \in \mathcal{B} \) and for \( 1 \leq m \leq n \)
\[ D_m S(t+h) \overset{\infty}{\approx} \sum_{i=1}^{\infty} U_i(t,h) | c_i(h), \quad \|h\| \to \infty, \quad h \in \Gamma. \]
If the family \( \{ V_i(t,h), i \in N, h \in \Gamma \} \) has the properties: \( D_m V_i(t,h) = U_i(t,h), \) \( i \in N, h \in \Gamma \) and for a \( \rho_o \in \mathcal{B}(R), \) \( \int \rho_o(t) dt = 1 \), and for every \( \rho \in \mathcal{B}, \) \( k \in N \)
\[ \lim_{\|h\| \to \infty, h \in \Gamma} \frac{< [S(t+h) - \sum_{i=1}^{k} V_i(t,h)], c_k(h), \rho_o(t) \lambda_m(t) >}{c_k(h) \rho_o(t) \lambda_m(t)} = 0 \]
where \( \lambda_m(t) = \int_0^t \rho(t_1, \ldots, t_m, \ldots, t_n) dt_m, \) then
\[ S(t+h) \overset{\infty}{\approx} \sum_{i=1}^{\infty} V_i(t,h) | c_i(h), \quad \|h\| \to \infty, \quad h \in \Gamma. \]

**PROOF.** If \( \rho \in \mathcal{B} \) then \( \rho(t) = \rho_o(t) \lambda_m(t) + \psi(t) \) where \( \psi \in \mathcal{B} \) and
\[ \int_{\mathbb{R}} \psi(t_1, \ldots, t_m, \ldots, t_n) dt_m = 0. \]
Now we have the following equality
\[ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \left< \left< \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \left< [S(t+h) - \sum_{i=1}^{k} V_i(t,h)], c_k(h), \rho(t) > \right> \right> \ldots \right> \right> = \left< \left< \left< \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \left[ D_m S(t+h) - \sum_{i=1}^{k} U_i(t+h) \right], \int_{\mathbb{R}} \psi(t_1, \ldots, t_m, \ldots, t_n) du_m > \right> \right> \ldots \right> \right> \right> \]
It remains only to use the limit in it and Corollary I.
PROPOSITION 5. Suppose that $S \in \mathcal{D}'$, $\Gamma = \{h \in \mathbb{R}^n, h = (0, \ldots, h_m, \ldots, 0)\}$, where $m$ is fixed, $1 \leq m \leq n$ and

$$(D_m S)(t+h) \overset{\|h\| \to \infty, h \in \Gamma}{\cong} \sum_{i=1}^{\infty} U_i(t,h) \mid \{c_i(h)\}, \quad \|h\| \to \infty, h \in \Gamma.$$ 

If there exists $V_i(t,h)$, $D_m V_i(t,h) = U_i(t,h)$, $i \in \mathbb{N}$ and if $c_i(h)$, $i \in \mathbb{N}$ are local integrable in $h_m$ and such that

$$c_i(h) \int_{h_m} c_i(u) \, du \to \infty \quad \text{as} \quad h_m \to \infty,$$

then

$$S(t+h) \overset{\|h\| \to \infty, h \in \Gamma}{\cong} \sum_{i=1}^{\infty} V_i(t,h) \mid \{\hat{c}_i(h)\}, \quad \|h\| \to \infty, h \in \Gamma.$$ 

PROOF. By L'Hospital's rule with the Stolz's improvement we have for every $\rho \in \mathcal{D}$ and $k \in \mathbb{N}$

$$\lim_{h \to \infty, h \in \Gamma} \frac{\langle S(t+h), \rho(t) \rangle - \sum_{i=1}^{k} V_i(t,h), \rho(t) \rangle}{\hat{c}_k(h)} = \lim_{h \to \infty, h \in \Gamma} \frac{\langle (D_m S)(t+h), \rho(t) \rangle - \sum_{i=1}^{k} U_i(t,h), \rho(t) \rangle}{c_k(h)}.$$

These five propositions give how is related the $S$-asymptotic with convolution, derivative, classical expansion and the primitive of a distribution. The next proposition gives the analytical expression of $U_n(t,h) U_n(t) c_n(h)$.

PROPOSITION 6. Suppose that $T \in \mathcal{D}'$, $\Gamma$ with nonempty interior,

$$T(t+h) \overset{\|h\| \to \infty, h \in \Gamma}{\cong} \sum_{n=1}^{\infty} U_n(t) c_n(h), \quad \|h\| \to \infty, h \in \Gamma.$$ 

If $u_m \neq 0$, $m \in \mathbb{N}$, then $u_m$ has the form

$$u_m(t) = \sum_{k=1}^{m} P_k^{m}(t_1, \ldots, t_n) \exp(a_k^m, t)$$

where $a_k = (a_k^1, \ldots, a_k^n) \in \mathbb{R}^n$ and $P_k^{m}$ are polynomials, the power of them less of $k$ in every $t_i$, $i = 1, \ldots, n$: $x, t \mapsto \prod_{i=1}^{n} x_i t_i$.

PROOF. By Definition 1 and our supposition

$$\lim_{h \to \infty, h \in \Gamma} T(t+h)/c_1(h) = u_1(t) \neq 0 \tag{3.5}$$

From relation (3.5) follows that $u_1$ satisfies the equation

$$u_1(t+h_0) = d(h_0) u_1(t), \quad h_0 \in \Gamma \quad \tag{3.6}$$

where

$$d(h_0) = \lim_{\|h\| \to \infty, h \in \Gamma} c_1(h+h_0)/c_1(h).$$
If $h_0$ is an interior point of $\Gamma$ and $e_k$ is such element from $\mathbb{R}^n$ for which all the coordinates equal zero except the $k$-th which is 1. Then

$$u_1(t+h_0+e_k) - u_1(t+h_0) = [d(e_k) - d(0)]u_1(t+h_0).$$

Hence the existence of $D_{h_0} d(h) = a^1_k$ and

$$D_{h_0} u_1(t+h_0) = a^1_k u_1(t+h_0), \quad k = 1, \ldots, n. \quad (3.7)$$

We know that all the solutions of equation (3.7) are of the form $u_1(t) = C_1 \exp(\langle a^1, t \rangle)$, where $C_1$ is a constant and $a^1 = (a^1_1, \ldots, a^1_n)$.

The following limit gives $u_2$

$$\lim_{\|h\| \to \infty, h \in \Gamma} \frac{\langle T(t+h), \rho(t) \rangle - \langle u_1(t), \rho(t) \rangle}{\|h\|} = c_2(h).$$

By Corollary follows for $i = 1, \ldots, n$

$$\lim_{\|h\| \to \infty, h \in \Gamma} \frac{\langle (D_{t_i} - a^1_i)T(t+h), \rho(t) \rangle}{\|h\|} = \frac{\langle (D_{t_i} - a^1_i)u_2(t), \rho(t) \rangle}{c_2(h)}.$$

Two cases are possible. a) If $(D_{t_i} - a^1_i)u_2 = 0$, $i = 1, \ldots, n$, then $u_2(t) = C_2 \exp(\langle a^1, t \rangle)$.

b) If $(D_{t_i} - a^1_i)u_2 \neq 0$ for some $i$, then $(D_{t_i} - a^1_i)u_2(t) = c \exp(\langle a^2, t \rangle)$ and $u_2$ has the form $C_2 \exp(\langle a^1, t \rangle) + P_2(t_1, \ldots, t_n) \exp(\langle a^2, t \rangle)$, where $P_2$ is a polynomial of the power less of 2 in every $t_i$, $i = 1, \ldots, n$.

In the same way we prove for every $u_m$.

**PROPOSITION 7.** Let $T \in \mathcal{F}$ and $\Omega \in \mathbb{R}^n$ be an open set with the property: for every $r > 0$ there exists a $\beta_r$ such that the ball $B(h, r) \subset \Omega$ for all $h \in \Gamma$, $\|h\| \geq \beta_r$.

Suppose

$$T(t+h) \supseteq \bigcup_{m=1}^m \left\{|c_1(h), \ldots, c_m(h)|, \|h\| \to \infty, h \in \Gamma \right\}$$

for any function $c_m(h)$ from $\Sigma(\Gamma)$, then $T = \bigcup_{m=1}^m \Omega_{n}$ over $\Omega$.

**PROOF.** The statement of this Proposition can be obtained from a proposition proved in [6]. However, for completeness, we shall give the proof on the whole.

First we shall prove that if for every $c_m(h) \in \Sigma(\Gamma)$

$$\lim_{\|h\| \to \infty, h \in \Gamma} \frac{\langle T(t+h) - \bigcup_{n=1}^m \Omega_n(t+h) \rangle}{\|h\|} = 0$$

then there exists a $\beta(\rho)$ such that

$$\langle [T(t+h) - \bigcup_{n=1}^m \Omega_n(t+h)], \rho(t) \rangle = 0, \quad h \in \Gamma, \quad \|h\| \geq \beta(\rho).$$

Suppose the opposite. We would have a sequence $h_n \in \Gamma$, $\|h_n\| \to \infty$ such that

$$\langle [T(t+h_n) - \bigcup_{n=1}^m \Omega_n(t+h_n)], \rho(t) \rangle = p_n \neq 0, \quad n \in N$$

then we choose $c_m(h)$ in such a way that $c_m(h_n) = p_n$ and relation (3.8) would be false.
We denote by $\beta_o(\rho) = \inf \beta(\rho)$. We shall prove that the set $\{\beta_o(\rho), \rho \in \mathcal{D}_k\}$ for every compact set $K \subset \mathbb{R}^n$ is bounded. Let us suppose the opposite; then there exists a sequence $\{h_k\}, h_k \in \Gamma$, $\|h_k\| \to +\infty$ and the sequence $\{\phi_k(t)\}, \phi_k \in \mathcal{D}_k$ such that

$$<\overline{T}(t+h_k), \phi_k(t)> = A_{k,p} = \begin{cases} a_k \neq 0, & p = k \\ 0, & p < k \end{cases}; \overline{T} = \Gamma - \sum_{n=1}^{m} U_n.$$

The construction of the sequence $\{h_k\}$ and $\phi_k$ can be the following. Let $\phi_k \in \mathcal{D}_k$ be such that $\beta_o(\phi_k)$ is a strict monotone sequence which tends to infinity, then there exist $\{h_k\} \subset \Gamma$ and $\varepsilon_k > 0, k \in \mathbb{N}$ such that $\beta_o(\phi_k+1) + \varepsilon_k \leq \|h_k\| \leq \beta_o(\phi_k) - \varepsilon_k$.

Now, we shall construct the sequence $\{\psi(t)\}, \psi \in \mathcal{D}_k$ for which we have

$$<\overline{T}(t+h_k), \psi(t)> = \begin{cases} 0, & p \neq k \\ a_k, & p = k \end{cases}.$$

Let $\psi(t) = \psi(t) = \phi(t) - \lambda^p_1 \phi(t) - \ldots - \lambda^p_{p-1} \phi(t), p > 1$. The numbers $\lambda^p_i$ we can find in such a way that $\psi(t)$ satisfies the sought property.

It is easy to see that $<\overline{T}(t+h_k), \phi_k(t)> = a_k$ and $<\overline{T}(t+h_k), \psi(t)> = 0, k > p$.

For a fixed $p$ and $k < p$ we can find $\lambda^p_i, i=1,\ldots,p-1$ so that for $k=1,\ldots,p-1$

$$0 = <\overline{T}(t+h_k), \psi(t)> = A_{k,p} - \lambda^p_1 A_{k,1} - \ldots - \lambda^p_{p-1} A_{k,p-1}.$$

Hence

$$\lambda^p_1 A_{k,1} + \ldots + \lambda^p_{p-1} A_{k,p-1} = A_{k,p}, k=1,\ldots,p-1, p > 1.$$

As $A_{k,k} \neq 0$ for every $k$, this system has always a solution.

We introduce now a sequence of numbers $\{b_k\}, b_k = \sup\{2^k |\psi^k(t)|, t < k\}$. Then the function

$$\psi(t) = \sum_{p=1}^{\infty} \psi(t)/b_{p} \in \mathcal{D}_K$$

and this series converges in $\mathcal{D}_k$, thus in $\mathcal{D}$ as well. With this

$$<\overline{T}(t+h_k), \psi(t)> = \sum_{p=1}^{\infty} <\overline{T}(t+h_k), \psi(t)/b_{p} > = a_k/b_k.$$

If we choose $c(h)$ such that $c(h_k) = a_k/b_k$ then $<\overline{T}(t+h)/c(h), \psi(t)>$ does not converge to zero when $\|h\| \to +\infty, h \in \Gamma$. This is in contradiction with (3.8). Hence, for every compact set $K$ there exists a $\beta_o(K)$ such that $<\overline{T}(t+h), \phi(t)> = 0, \|h\| \geq \beta_o(K), h \in \Gamma, \phi \in \mathcal{D}_k$. That means that $\overline{T}(t+h) = 0$ over $B(0,r), \|h\| \geq \beta(r), h \in \Gamma$ and $\overline{T}(t) = 0$ over $B(h,r), \|h\| \geq \beta(r), h \in \Gamma$.

4. APPLICATION OF THE S-ASYMPTOTIC EXPANSION TO PARTIAL DIFFERENTIAL EQUATIONS.

As we mentioned in [4], one can find cited literature in which asymptotic expansion technique (in $\mathcal{D}$ and in one dimensional case) was used in the quantum field theory. We show how the S-asymptotic expansion in $\mathcal{D}$ can be applied to solutions of partial differential equations.

PROPOSITION 8. Suppose that $E$ is a fundamental solution of the operator

$$L(D) = \sum_{|a| \geq 0} a^a D^a, a^a \in \mathbb{R}, a \in (N\mathbb{U})^n; L(D) \neq 0 \quad (4.1)$$

such that
Then there exists a solution $X$ of the equation

$$L(D)X = G, \ G \in \mathcal{E}$$

which has $S$-asymptotic expansion

$$X(t+h) \overset{\mathcal{S}}{=} \sum_{n=1}^{\infty} (G \ast u_n(t,h)) \{c_n(h)\}, \ ||h|| \to \infty, \ h \in \Gamma.$$  

PROOF. The well-known Malgrange-Ehrenpreis theorem (see for example [7], p. 212) asserts that there exists a fundamental solution of the operator (4.1) belonging to $\mathcal{G}^\prime$. The solution of equation (4.3) exists and can be expressed by the formula $X = E \ast G$. To find the $S$-asymptotic of $X$ we have only to apply Proposition 1.

REMARKS. If we denote by $A(L(D),E)$ the collection of those $T \in \mathcal{G}^\prime$ for which the convolution $E \ast T$ and $L(D)\delta \ast E \ast T$ exist in $\mathcal{G}^\prime$, then the solution $X = E \ast G$ is unique in the class $A(L(D),E)$ ([7], p. 87).

We can enlarge the space to which belongs $G$ ([7], p. 216).

The fundamental solutions are known for the most important operators $L(D)$.

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