SOME RESULTS CONCERNING EXPONENTIAL DIVISORS

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Abstract. If the natural number $n$ has the canonical form $p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ then $d = p_1^{b_1}p_2^{b_2}...p_r^{b_r}$ is said to be an exponential divisor of $n$ if $b_i | a_i$ for $i = 1, 2, ..., r$. The sum of the exponential divisors of $n$ is denoted by $\sigma(e)(n)$. $n$ is said to be an e-perfect number if $\sigma(e)(n) = 2n$. $(m; n)$ is said to be an e-amicable pair if $\sigma(e)(m) = m + n = \sigma(e)(n)$; $n_1, n_2, ...$ is said to be an e-aliquot sequence if $n_{i+1} = \sigma(e)(n_i) - n_i$. Among the results established in this paper are: the density of the e-perfect numbers is .0087; each of the first 10,000,000 e-aliquot sequences is bounded.

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1. INTRODUCTION.

If $n$ is a positive integer greater than one whose prime-power decomposition is given by

$$n = p_1^{a_1}p_2^{a_2}...p_r^{a_r}$$

(1.1)

then $d$ is said to be an "exponential divisor" of $n$ if $d = p_1^{b_1}p_2^{b_2}...p_r^{b_r}$ where $b_i | a_i$ for $i = 1, 2, ..., r$. The sum of all of the exponential divisors of $n$ is denoted by $\sigma(e)(n)$. This function was first studied by Subbarao [1] who also initiated the study of exponentially perfect (or e-perfect) numbers.

The positive integer $n$ is said to be an e-perfect number if $\sigma(e)(n) = 2n$. If $\sigma(e)(n) = kn$, where $k$ is an integer which exceeds 2, $n$ is said to be an e-multi-perfect number. The properties of e-perfect and e-multiperfect numbers have been investigated by Straus and Subbarao [2] and Fabrykowski and Subbarao [3]. It has been proved, for example, that all e-perfect and e-multiperfect numbers are even. Also, if $n$ is an e-perfect number and $3 | n$ then $2^{110} | n$ and $n > 10^{618}$.

While it is easy to show that there are an infinite number of e-perfect numbers, whether or not any e-multiperfect numbers exist is still an open question. Subbarao, Hardy and Aiello [4] have conjectured that there are no e-multiperfect numbers. They have proved that any which exist are very large.
In Section 2 of the present paper the density of the set of \( e \)-perfect numbers is investigated. Section 3 is devoted to a study of \( e \)-amicable pairs, integers \( m \) and \( n \) such that \( o^{(e)}(m) = m + n = o^{(e)}(n) \). Finally, \( e \)-aliquot sequences \( n_0, n_1, n_2, \ldots \) where \( n_{i+1} = o^{(e)}(n_i) - n_i \) for \( i = 0, 1, 2, \ldots \) are studied in Section 4.

2. THE DENSITY OF THE \( e \)-PERFECT NUMBERS.

By definition, \( o^{(e)}(1) = 1 \) and it is easy to see that \( o^{(e)}(n) \) is multiplicative. Therefore, since \( o^{(e)}(p) = p \) if \( p \) is a prime, we see that \( o^{(e)}(m) = m \) if \( m \) is square-free.

Now suppose that \( n \), as given by (1.1), is a powerful \( e \)-perfect number (so that \( a_i \geq 2 \) for \( i = 1, 2, \ldots, r \) and \( o^{(e)}(n) = 2n \)). Then if \( (m, n) = 1 \) and \( m \) is squarefree then \( o^{(e)}(mn) = 2mn \) so that \( mn \) is an \( e \)-perfect number. Therefore, if \( x \) is a (fixed) positive number and \( n_1 < n_2 < \ldots < n_s \) are the powerful \( e \)-perfect numbers which do not exceed \( x \) then \( E(x) \), the set of (all) \( e \)-perfect numbers less than or equal to \( x \), is given by

\[
E(x) = \bigcup_{i=1}^{s} A_i \text{ where } A_i = \{mn_i : (m, n_i) = 1, m \leq x/n_i \text{ and } m \text{ is squarefree} \}
\]

Let \( N \) be a positive integer and let \( X \) be a positive real number. If \( Q(N, X) \) is the number of positive, squarefree integers which do not exceed \( X \) and which are relatively prime to \( N \), then E. Cohen (Lemma 5.2 in [5]) has shown that

\[
Q(N, X) = \beta(N) \cdot X + O(\theta(N) \cdot X^{1/2})
\]

where \( \beta(N) = (\zeta(2) \prod_{p \mid N} (1 + 1/p))^{-1} \) and \( \theta(N) \) is the number of squarefree divisors of \( N \). It is easy to see that \( \theta(N) = \prod_{p \mid N} 2 \). \( \zeta(k) \) is the Riemann Zeta function, so that \( \zeta(2) = \pi^2/6 \), and the constant implied by the \( O \)-term is independent of \( N \) and \( X \).

If \( Q(e, x) \) is the number of \( e \)-perfect numbers which do not exceed \( x \) (so that \( Q(e, x) \) is the cardinality of \( E(x) \)) it follows from (2.1) and (2.2) that

\[
Q(e, x) = x \sum_{i=1}^{s} \beta(n_i) / n_i + O(x^{1/2} \sum_{i=1}^{s} \theta(n_i) / n_i^{1/2}).
\]

Therefore,

\[
Q(e, x) / x = \sum_{i=1}^{s} \beta(n_i) / n_i + O(x^{-1/2} \sum_{i=1}^{s} \theta(n_i) / n_i^{1/2}).
\]

The following results concerning powerful numbers will be needed in what follows.

Proofs may be found in Golomb [6].

**LEMMA 1.** If \( r_1 < r_2 < \ldots \) is the sequence of powerful numbers then \( \sum_{i=1}^{\infty} 1/r_i \) is convergent.

**LEMMA 2.** If \( P(X) \) is the number of powerful numbers not exceeding \( x \) then \( P(x) < 2.2x^{1/2} \) for large \( x \).

Now let \( e \) be a given positive number and let \( P_i \) denote the \( i \)th prime. There exists a positive integer \( k \) such that

\[
2/P_k < e \cdot (2.2K)^{-1/3}
\]

where \( K \) is the constant implied by the \( O \)-term in (2.3).
Since there are only a finite number of powerful e-perfect numbers which are divisible by fewer than \( k \) distinct primes (see Theorem 2.3 in [2]) there exists a positive integer \( J \) such that if \( n_1 < n_2 < \ldots \) is the sequence of powerful e-perfect numbers then for all \( i > J \) \( n_i \) has at least \( k \) distinct prime factors and \( n_i \) has a prime factor, say \( q_i \), such that \( q_i \geq P_{\sqrt{k}} \). Since \( n_i \) is powerful, \( n_i^{1/2} \geq \Pi p \) where the product is taken over the distinct prime factors of \( n_i \), and it follows from (2.4) that if \( i > J \) then

\[
\frac{\sigma(n_i)/n_{1/2}}{\Pi p < 2/\sqrt{\sqrt{k}} < \epsilon \cdot (2.2k)^{-1/3}.}
\]

(2.5)

Splitting the sum in the O-term in (2.3) at \( i = J \) (with \( J \) held fixed) we can take \( x \) large enough so that \( x^{-1/2} \cdot \sum_{i=1}^{J} \frac{\sigma(n_i)/n_{1/2}}{\Pi p} < \epsilon \cdot (2.2k)^{-1/3}. \) At the same time, since every \( n_i \) is powerful, we see from (2.5) and Lemma 2 that we can also take \( x \) large enough so that

\[
x^{-1/2} \cdot \sum_{i=J+1}^{\infty} \frac{\sigma(n_i)/n_{1/2}}{\Pi p} < \epsilon \cdot (2.2k)^{-1/3}
\]

Finally, since \( \sigma(n_i) < 1 \) and every \( n_i \) is powerful we see from Lemma 1 that

\[
\sum_{i=1}^{\infty} \frac{\sigma(n_i)/n_{1/2}}{\Pi p} \text{ is convergent. (This series may be finite since whether or not the set of powerful e-perfect numbers is finite or infinite is an open question). It follows that we can take } x \text{ (and consequently } s) \text{ large enough so that the tail of this series is less than } \epsilon / 3. \text{ Therefore, from (2.3) we have for all large values of } x,
\]

\[
\left| \frac{Q(e,x)/x - \sum_{i=1}^{\infty} \frac{\sigma(n_i)/n_{1/2}}{\Pi p}}{\epsilon} \right| < \epsilon
\]

(2.6)

We have proved

THEOREM 1. Let \( Q(e,x) \) denote the number of e-perfect numbers which do not exceed \( x \) and let \( n_1 < n_2 < n_3 < \ldots \) be the sequence of powerful numbers. Then

\[
\lim_{x \to \infty} \frac{Q(e,x)/x}{\sum_{i=1}^{\infty} \frac{\sigma(n_i)/n_{1/2}}{\Pi p}} = C
\]

where \( C = 6\pi^{-2} \sum_{p} (1+1/p)^{-1} \). Correct to ten decimal places, \( C = .0086941940 \).

(There are eight powerful e-perfect numbers less than \( 10^{10} \): 36; 1800; 2700; 17,424; 1,306,800; 4,769,856; 238,492,800; 357,739,200. The approximate value of \( C \) given above was calculated using these eight numbers).

The "theoretical" density of the e-perfect numbers as given in Theorem 1 agrees very nicely with the following exact computational results: \( Q(e,10^5)/10^5 = .00871 \); \( Q(e,10^6)/10^6 = .008690 \); \( Q(e,10^7)/10^7 = .0086940 \); \( Q(e,10^8)/10^8 = .00869417 \).

3. EXPONENTIALLY AMICABLE NUMBERS.

We shall say that \( m \) and \( n \) are exponentially amicable (or e-amicable) numbers if

\[
\sigma(e)(m) = m + n = \sigma(e)(n).
\]

(3.1)
LEMMA 3. If \((m;n)\) is an e-amicable pair and \(p\) is a prime, then \(p|m\) if and only if \(p|n\).

PROOF. Suppose that \(p^{a}|m\) where \(a \geq 1\). Then \(p|o(e)(m)\) since \(p|o(e)(p^{a})\) and \(o(e)\) is a multiplicative function. It is now obvious from (3.1) that \(p|m\). By the same argument, if \(p|n\) then \(p|m\).

COROLLARY 3.1. If \((m;n)\) is an e-amicable pair then \(m \equiv n \pmod{2}\).

If \((m;n)\) is an e-amicable pair and there is no prime \(p\) such that \(p|\frac{m}{n}\) we shall say that \(m\) and \(n\) are primitive e-amicable numbers. It is easy to see that if \((m;n)\) is a primitive e-amicable pair and \(r\) is a squarefree positive integer such that \((m,r) = 1\), then \((rm;rn)\) is an amicable pair.

A search was made for all primitive e-amicable pairs \((m;n)\) such that \(m < n\) and \(m < 10^7\). The search required about 1.5 hours on the CDC CYBER 750 and three pairs were found. They are as follows: \((2^{3 \cdot 2^2} \cdot 3^2 \cdot 219^2; 2^{3 \cdot 2^2} \cdot 3^2 \cdot 219^2)\); \((2^{3 \cdot 2^2} \cdot 61^2; 2^{3 \cdot 2^2} \cdot 61^2)\); \((2^{3 \cdot 2^2} \cdot 57^2 \cdot 19^2; 2^{3 \cdot 2^2} \cdot 57^2 \cdot 19^2)\).

This list suggests the following questions. Are there any odd e-amicable numbers? Are there any powerful e-amicable numbers? Is every e-amicable number divisible by at least four distinct primes? (It is easy to show that every e-amicable number has at least three different prime factors).

The following result can sometimes be used to generate new e-amicable pairs from known pairs.

THEOREM 2. Suppose that \((aM;aN)\) is an e-amicable pair such that \((a,M) = (a,N) = 1\). If \((b,M) = (b,N) = 1\) and \(o(e)(a)/a = o(e)(b)/b\) then \((bM,bN)\) is an e-amicable pair.

PROOF. \(o(e)(bM) = o(e)(b) \cdot o(e)(M) = a^{-1} b o(e)(a) \cdot o(e)(M) = a^{-1} b o(e)(aM) = a^{-1} b(aM +aN) = bM +bN\). Similarly, \(o(e)(bN) = bM +bN\).

The results of a computer search for powerful numbers \(a\) and \(b\) such that \(4 \leq a < b \leq 10000\) and \(o(e)(a)/a = o(e)(b)/b\) are given in Table I.

<table>
<thead>
<tr>
<th>(o(e)(a)/a)</th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/2</td>
<td>(2^2)</td>
<td>(2^3 2^2) or (2^4 11^2)</td>
</tr>
<tr>
<td>4/3</td>
<td>(3^2)</td>
<td>(3^3 2^2)</td>
</tr>
<tr>
<td>2</td>
<td>(2^2 3^2)</td>
<td>(2^3 5^2 2^2) or (2^2 3^2 5^2)</td>
</tr>
<tr>
<td>39/32</td>
<td>(6^2)</td>
<td>(2^7 2^2)</td>
</tr>
<tr>
<td>5/3</td>
<td>(2^3 3^2)</td>
<td>(2^2 3^3) or (2^2 3^3 3^2)</td>
</tr>
<tr>
<td>12/7</td>
<td>(2^7 2^2)</td>
<td>(2^3 5^2 2^2)</td>
</tr>
<tr>
<td>65/48</td>
<td>(2^7 3^2)</td>
<td>(2^6 3^2)</td>
</tr>
<tr>
<td>40/21</td>
<td>(2^3 3^2 7^2)</td>
<td>(2^3 3^2 7^2)</td>
</tr>
</tbody>
</table>
EXAMPLE. Since \((2^2 \cdot 3^2 \cdot 7 \cdot 19^2; 2^3 \cdot 3^2 \cdot 7^2 \cdot 19)\) is an \(e\)-amicable pair and since \(s(e)(2^2) = s(e)(3^2) = s(e)(7^2) = s(e)(19^2) = 1\), it follows from Theorem 2 that \((2^2 \cdot 19^2; 2^3 \cdot 3^2 \cdot 7^2 \cdot 19)\) is an \(e\)-amicable pair.

4. EXPONENTIAL ALIQUOT SEQUENCES.

The function \(s(e)\) is defined by \(s(e)(n) = s(e)(n) - n\), the sum of the exponential aliquot divisors of \(n\). \(s(e)(1) = s(e)(r) = 0\) for every squarefree number \(r\) and we define \(s(e)(0) = 0\). A \(t\)-tuple of distinct natural numbers \((n_0, n_1, \ldots, n_{t-1})\) with \(n_i = s(e)(n_{i-1})\) for \(i = 1, 2, \ldots, t-1\) and \(s(e)(n_{t-1}) = n_0\) is called an exponential \(t\)-cycle. An exponential 1-cycle is an \(e\)-perfect number and an exponential 2-cycle is an \(e\)-amicable pair. A search was made for all exponential \(t\)-cycles with smallest member not exceeding \(10^7\). None with \(t > 2\) was found.

The exponential aliquot sequence (or \(e\)-aliquot sequence) \((n_i)\) with leader \(n\) is defined by \(n_0 = n, n_1 = s(e)(n_0), n_2 = s(e)(n_1), \ldots\). Such a sequence is said to be terminating if \(n_k\) is squarefree for some index \(k\) (so that \(n_i = 0\) for \(i > k\)). An exponential aliquot sequence is said to be periodic if there is an index \(k\) such that \((n_k; n_{k+1}; \ldots; n_{k+t-1})\) is an exponential \(t\)-cycle. An \(e\)-aliquot sequence which is neither terminating nor periodic is unbounded.

An investigation was made of all aliquot sequences with leader \(n \leq 10^7\). About 23 hours of computer time was required. 9,896,235 were found to be terminating and 103,765 were periodic (103,694 ended in 1-cycles and 71 ended in 2-cycles).

The fact that the first ten million exponential aliquot sequences are bounded might tempt one to conjecture that the set of unbounded \(e\)-aliquot sequences is empty. However, the following theorem shows that \(e\)-aliquot sequences exist which contain arbitrarily long strings of monotonically increasing terms. Therefore, whether or not unbounded \(e\)-aliquot sequences exist would seem to be a very open and difficult question.

THEOREM 3. Let \(N\) be a positive integer which exceeds 2. Then there exist infinitely many exponential aliquot sequences such that \(n_0 < n_1 < n_2 < \ldots < n_N < 2\).

PROOF. Let \(q_1, q_2, \ldots, q_N\) be a sequence of \(N\) primes such that \(q_1 = 2, q_2 = 3\) and \(q_i^2 | (q_{i+1} + 1)\) for \(i = 2, 3, \ldots, N-1\). (Infinitely many such sequences exist since, by Dirichlet's theorem, the arithmetic progression \(aq_1^2 - 1\) contains an infinite number of primes.) We shall write \(q_i^2 + 1 = K_i q_1^2\).

Now let \(n_0, n_1, n_2, \ldots\) be the exponential aliquot sequence with leader \(n_0\) given by \(n_0 = q_1^2 q_2^2 \ldots q_N^2\). Then

\[
s(e)(n_0) = \prod_{i=1}^{N} \left( q_1^2 + q_i^2 \right) = 3 \cdot q_1 q_2 \cdots q_N \cdot \prod_{i=2}^{N} (1 + q_i)
\]

\[
= 3 \cdot q_1 q_2 \cdots q_N \cdot \prod_{i=1}^{N-1} K_i q_1^2,
\]

and

\[
n_1 = s(e)(n_0) - n_0 = (3 \cdot q_1 q_2 \cdots q_N \cdot K_1 \cdots K_{N-1} - q_1^2) \cdot \prod_{i=1}^{N-1} q_i^2.
\]

Therefore, \(n_1 = M_1 \prod_{i=1}^{N-1} q_i^2\) where \((M_1, q_i) = 1\) for \(i = 1, 2, \ldots, N-1\).
Since $n_0/36$ is not squarefree, $n_1 = \sigma(e)(n_0) - n_0 = \sigma(e) (36) \cdot \sigma(e)(n_0/36) - n_0 = 72 - \sigma(e)(n_0/36) - n_0 > 72 \cdot n_0/36 - n_0 = n_0$.

Similarly, we find that for $k = 2, 3, \ldots, N-2$

$$n_k = M_k \prod_{i=1}^{N-k} \frac{q_i}{n_i}^2 \text{ where } (M_k, q_i) = 1 \text{ for } i = 1, 2, \ldots, N-k$$

and

$$n_k = \sigma(e)(n_{k-1}) - n_{k-1} = \sigma(e)(36) \cdot \sigma(e)(n_{k-1}/36) - n_{k-1} > 72 \cdot n_{k-1}/36 - n_{k-1} = n_{k-1}.$$

Therefore, $n_0 < n_1 < \ldots < n_{N-2}$.

**Remark 1.** $n_{N-2} = 36M_{N-2}$ where $(6, M_{N-2}) = 1$. If $M_{N-2}$ is not squarefree, then $n_{N-1} = 72 \cdot \sigma(e)(M_{N-2})/36 > 72M_{N-2}/36 > 36M_{N-2} = 36n_{N-2}$.

**Remark 2.** The proof of Theorem 3 is modeled on that of Theorem 2.1 in [7].

Our next objective is to determine $M(\sigma(e)(n)/n)$, the mean value of $\sigma(e)(n)/n$.

The mean value of an arithmetic function $f$ is defined by $M(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n)$.

We shall need the following lemma due to van der Corput (See Theorem A in [8].)

**Lemma 4.** If $f$ and $h$ are arithmetic functions such that $f(n) = h(d)$ and $\sum_{n=1}^{\infty} h(n)/n$ is absolutely convergent then $M(f) = \sum_{d|n} h(d)$.

We wish to apply this lemma to the function $f(n) = \sigma(e)(n)/n$. By the Möbius inversion formula, $h(n) = \sum_{d|n} \mu(n/d) \sigma(e)(d)/d$. $h$ is multiplicative and $h(1) = 1$.

If $p$ is a prime and $a$ is a positive integer then $h(p^a) = \sigma(e)(p^a)/p^a = \sigma(e)(p^{a-1})/p^{a-1}$. If $a < 6$ it is easy to verify that $|h(p^a)| < p^{a-4}$ (For example, $|h(p^3)| = p^{a-1} - p^{-2} < p^{-1} < p^{-3/4}$.) Suppose that $a \geq 6$. Then $|h(p^a)| = \sigma(e)(p^a)/p^{a-1} \sigma(e)(p^{a-1})/p^{a-1}$ or $|h(p^a)| = \sigma(e)(p^{a-1})/p^{a-1} \sigma(e)(p^a)/p^a$.

Since $\sigma(e)(p^m)/p^m < 1 + 1/(p-1)p^{m/2}$ (see [2] or [4]) and $\sigma(e)(p^b)/p^b \geq 1$, $|h(p^a)| < p/(p-1)(p^{a-1}/2)$. Since $a \geq 6$ it follows easily that $|h(p^a)| < p^{-a/4}$.

Since $h$ is multiplicative, $|h(n)| \leq n^{-1/4}$ for every positive integer $n$. It follows that $\sum_{n=1}^{\infty} h(n)/n$ is absolutely convergent so that Lemma 4 applies if $f(n) = \sigma(e)(n)/n$.

From Theorem 286 in [9] we have

$$\sum_{n=1}^{\infty} h(n)/n = \prod_{p} \left(1 + h(p)/p + h(p^2)/p^2 + \ldots\right)$$

$$= \prod_{p} \left(1 + p^{-1}(\sigma(e)(p)/p-1) + p^{-2}(\sigma(e)(p^2)/p^2 - \sigma(e)(p)/p) + \ldots\right)$$

$$= \prod_{p} \left(\sum_{j=0}^{\infty} \sigma(e)(p^j)/p^{2j} - p^{-1} \sum_{j=0}^{\infty} \sigma(e)(p^j)/p^{2j}\right)$$

$$= \prod_{p} \left((1 - p^{-1}) \sum_{j=0}^{\infty} \sigma(e)(p^j)/p^{2j}\right).$$

Now the last infinite series can be "split up" by first taking all the terms with numerator $p^j$ to form the series $\sum_{j=0}^{\infty} p^j/p^{2j} = \sum_{j=0}^{\infty} 1/p^{j}$; then taking the remaining
terms with numerators $p$ to form the series $\sum_{j=2}^{\infty} \frac{p/p^{2j}}{p^{2j-3}}$; then taking
the terms with numerators $p^2$ to form the series $\sum_{j=2}^{\infty} \frac{p^2/p^{4j}}{p^{4j-6}}$; then
taking the terms with numerators $p^3$ to form the series $\sum_{j=2}^{\infty} \frac{p^3/p^{6j}}{p^{6j-9}}$; etc. It follows that

$$\sum_{n=1}^{\infty} \frac{h(n)/n}{p} = \prod (1 - p^{-1})(1 - p^{-1})^{-1} + p^{-3}(1 - p^{-2})^{-1}$$

$$+ p^{-6}(1 - p^{-4})^{-1} + p^{-9}(1 - p^{-6})^{-1} + \ldots)$$

$$= \prod \left( (1 - p^{-1})(1 - p^{-1})^{-1} + (p^3 - p)^{-1} + (p^6 - p^2)^{-1} \right)$$

$$+ (p^9 - p^3)^{-1} + \ldots)$$

$$= \prod \left( 1 + (1 - p^{-1}) \sum_{j=1}^{\infty} (p^{3j} - p^j)^{-1} \right).$$

From Lemma 4 we have

**THEOREM 4.** $\lim_{n \to \infty} \frac{s(e)(n)/n}{p} = \prod \left( 1 + (1 - p^{-1}) \sum_{j=1}^{\infty} (p^{3j} - p^j)^{-1} \right) = \frac{1}{136571}.$

Correct to 6 decimal places, $C = 1.136571$.

(This approximate value of $C$ was calculated using all primes less than $10^6$ in
the infinite product.)

Since $s(e)(n) = o(e)(n) = n$ we have

**COROLLARY 4.1.** $\lim_{n \to \infty} \frac{s(e)(n)/n}{p} = \frac{1}{136571}.$

Finally, since $n_{i+1}/n_i = s(e)(n_i)/n_i$, we see that, in some sense, the average
value of the ratio of two consecutive non-zero terms of an e-aliquot sequence is
about $1.136571$.

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