ABSTRACT. It is shown that inversion is a convex function on the set of strictly positive elements of a C*-algebra.

KEY WORDS AND PHRASES. Convex function, C*-algebra.

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1. INTRODUCTION.

A real-valued function $f$ defined on a real interval $I$ is said to be convex if

$$f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda)f(t)$$

for $s,t \in I$ and $0 \leq \lambda \leq 1$. Convex functions play a fundamental role in the study of the Lebesgue $L^p$ spaces [1], [2]. Geometrically, a function $f$ is convex if the chord joining the points $(s, f(s))$ and $(t, f(t))$ lies above the graph of $f$. An interesting example of a convex function is the function $f(t) = t^{-1}$, $t \in I = (0, \infty)$. Thus inversion is a convex function on the set of positive reals. The notion of convexity has been generalized to functions with domain and range more general than reals. For instance, through a diagonalization process it is shown in [3] that inversion is a convex function on the set of positive-definite real symmetric matrices. In this note we will show that this result holds in a C*-algebra. More precisely, we use Banach algebra techniques to show that inversion is a convex function on the set of strictly positive elements of a C*-algebra.

2. PRELIMINARIES.

Throughout this article $\mathcal{A}$ will denote a complex C*-algebra with identity $e$. An element $x \in \mathcal{A}$ is said to be self-adjoint if $x^* = x$, where $x^*$ is the adjoint of $x$. A self-adjoint element $x$ is said to be non-negative, in notation $x \geq 0$, if its spectrum $\sigma(x)$ lies in the interval $[0, \infty)$. For self-adjoint elements $x$ and $y$, we write $x \leq y$ if $y - x \geq 0$. An element $x$ will be termed strictly positive if it is non-negative and invertible. Thus $x$ is strictly positive if $x$ is self-adjoint and $\sigma(x)$ lies in the interval $(0, \infty)$. If $x$ is an invertible element then we use $x^{-1}$ to denote its inverse.

A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is said to be self-adjoint if $x \in \mathcal{B}$ implies $x^* \in \mathcal{B}$. The main tools we need to establish our result are:

(A) If $\mathcal{B}$ is a closed self-adjoint subalgebra of $\mathcal{A}$ and $x \in \mathcal{B}$, then $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$. Here $\sigma_{\mathcal{B}}(x)$ and $\sigma_{\mathcal{A}}(x)$ denote the spectra of $x$ relative to $\mathcal{B}$ and $\mathcal{A}$, respectively.
(B) If \( \mathcal{B} \) is a commutative Banach algebra and \( x \in \mathcal{B} \) then \( \sigma_{\mathcal{B}}(x) = (\varphi(x)) \varphi \) a complex homomorphism on \( \mathcal{B} \).

Proofs of (A) and (B) may be found, for example, in [4].

3. MAIN RESULT.

**Lemma:** If \( u \) is a strictly positive element of \( \mathcal{A} \), then
\[
[\lambda e + (1 - \lambda) w]^{-1} \leq \lambda e + (1 - \lambda) w^{-1} \quad \text{for} \quad 0 \leq \lambda \leq 1.
\]

**Proof:** Let \( \mathcal{B} = \mathcal{B}(w,e) \) be the closed subalgebra generated by \( w \) and \( e \). Since \( w \) is self-adjoint, \( \mathcal{B} \) is self-adjoint and commutative. Clearly \( w \) and \( \lambda e + (1 - \lambda) w \) are elements of \( \mathcal{B} \). Since these elements are invertible in \( \mathcal{A} \), \( u = \lambda e + (1 - \lambda) w \) and \( v = \lambda e + (1 - \lambda) w^{-1} \) are elements of \( \mathcal{B} \) by (A). Our goal is to show that \( \sigma_{\mathcal{B}}(u - v) \) lies in \( [0, \infty) \). In view of (B) it suffices to show that \( \varphi(u) \leq \varphi(v) \) for complex homomorphisms \( \varphi \) on \( \mathcal{B} \). Since \( \varphi(u) = \lambda + (1 - \lambda) \varphi(w)^{-1} \) and \( \varphi(v) = \lambda + (1 - \lambda) \varphi(w) \), the result follows from the fact that \( f(t) = t^{-1} \) is a convex function on \((0, \infty)\).

**Theorem:** If \( x \) and \( y \) are strictly positive elements of \( \mathcal{A} \), then
\[
[\lambda x + (1 - \lambda) y]^{-1} \leq \lambda x^{-1} + (1 - \lambda) y^{-1} \quad \text{for} \quad 0 \leq \lambda \leq 1.
\]

**Proof:** First we recall that if \( p \) and \( q \) are self-adjoint elements of \( \mathcal{A} \) with \( p \leq q \), then \( r^*pr \leq r^*qr \) for any \( r \in \mathcal{A} \). This fact from C*-algebra theory will be used twice in the proof.

Now, since \( x \) is strictly positive, it possesses a unique strictly positive square root, say \( z \), in \( \mathcal{A} \). Then \( w = z^{-1} y z^{-1} \) is strictly positive. By the lemma, we have
\[
[\lambda e + (1 - \lambda) w]^{-1} \leq \lambda e + (1 - \lambda) w^{-1}
\]

Thus
\[
z^{-1} [\lambda e + (1 - \lambda) w]^{-1} z^{-1} \leq z^{-1} [\lambda e + (1 - \lambda) w] z^{-1}
\]

This in turn gives
\[
[\lambda x + (1 - \lambda) y]^{-1} \leq \lambda x^{-1} + (1 - \lambda) y^{-1}
\]

The proof is thus complete.

**REFERENCES**


