TWO PROPERTIES OF THE POWER SERIES RING

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ABSTRACT. For a commutative ring with unity, A, it is proved that the power series ring \( A[[X]] \) is a PF-ring if and only if for any two countable subsets \( S \) and \( T \) of \( A \) such that \( S \subseteq \text{ann}(T) \), there exists \( c \in \text{ann}(T) \) such that \( bc = b \) for all \( b \in S \). Also it is proved that a power series ring \( A[[X]] \) is a PP-ring if and only if \( A \) is a PP-ring in which every increasing chain of idempotents in \( A \) has a supremum which is an idempotent.

KEY WORDS AND PHRASES. Power series ring, PP-ring, PF-ring, flat, projective, annihilator ideal and idempotent element.

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1. INTRODUCTION.

Rings considered in this paper are all commutative with unity. Let \( A[[X]] \) be the power series ring over the ring \( A \). Recall that a ring \( A \) is called a PF-ring if every principal ideal is a flat \( A \)-module. Also a ring \( A \) is called a PP-ring if every principal ideal is a projective \( A \)-module.

It is proved in Al-Ezeh [1] that a ring \( A \) is a PF-ring if and only if the annihilator of each element \( a \in A \), \( \text{ann}(a) \), is a pure ideal, that is for all \( b \in \text{ann}(a) \) there exists \( c \in \text{ann}(a) \) such that \( bc = b \). A ring \( A \) is a PP-ring if and only if for each \( a \in A \), \( \text{ann}(a) \) is generated by an idempotent, see Evans [2]. In Brewer [3], semihereditary power series rings over von Neumann regular rings are characterized. In this paper we characterize PF- power series rings and PP- power series rings over arbitrary rings.

For any reduced ring \( A \) (i.e. a ring with no nonzero nilpotent elements), it was proved in Brewer et al. [4] that

\[
\text{ann} (a_0 + a_1X + \ldots) = N[[X]]
\]

where \( N \) is the annihilator of the ideal generated by the coefficients \( a_0, a_1, \ldots \).

2. MAIN RESULTS.
LEMMA 1. Any PF-ring \( A \) is a reduced ring.

PROOF. Assume that there is a nonzero nilpotent element in \( A \). Let \( n \) be the least positive integer greater than 1 such that \( a^n = 0 \). So \( a \in \text{ann}(a^{n-1}) \). Because \( A \) is a PF-ring there exists \( b \in \text{ann}(a^{n-1}) \) such that \( ab = a \). Thus \( a^{n-1}A = (ab)^{n-1} = a^{n-1}b^{n-1} = 0 \) since \( ba^{n-1} = 0 \).
Contradiction. So any PF-ring is a reduced ring.

THEOREM 2. The power series ring \( A[[X]] \) is a PF-ring if and only if for any two countable sets \( S = \{b_0, b_1, b_2, \ldots \} \) and \( T = \{a_0, a_1, \ldots \} \) such that \( S \subseteq \text{ann}(T) \), there exists \( c \in \text{ann}(T) \) such that \( b_i c = b_i \) for \( i = 0, 1, 2, \ldots \)

PROOF. First, we prove that \( A[[X]] \) is a PF-ring.

Let \( g(X) = b_0 + b_1X + \ldots \), and
\[
g(X) \in \text{ann} \ (f(X)).
\]
Then \( g(X) f(X) = 0 \).

The ring \( A \) is in particular a PF-ring because for all \( b \in \text{ann}(a) \), there exists \( c \in \text{ann}(a) \) such that \( bc = b \). So by Lemma 1, \( A \) is a reduced ring. Thus

\[
b_i a_j = \text{for all } i = 0, 1, \ldots; j = 0, 1, 2, \ldots
\]

So
\[
\{b_0, b_1, \ldots \} \subseteq \text{ann}(a_0, a_1, \ldots).
\]

So by assumption, there exists \( c \in \text{ann}(a_0, a_1, \ldots) \) such that \( b_i c = b_i \) for all \( i = 0, 1, \ldots \). Hence \( g(X)c = g(X) \)

and \( c \in \text{ann} \ (f(X)) \). Consequently, the ring \( A[[X]] \) is a PF-ring. Conversely, assume \( A[[X]] \) is a PF-ring.

Let \( \{b_0, b_1, \ldots \} \subseteq \text{ann}(a_0, a_1, \ldots) \). Let \( g(X) = b_0 + b_1X + \ldots \), and \( f(X) = a_0 + a_1X + \ldots \).

Then \( g(X) f(X) = 0 \). Therefore \( g(X) c \in \text{ann} \ (f(X)) \). Thus there exists \( h(X) = c_0 + c_1X + \ldots \)

in \( \text{ann} \ (f(X)) \) such that \( g(X) h(X) = g(X) \).

Consequently, \( h(X) f(X) = 0 \) and \( g(X) (h(X) - 1) = 0 \). Since \( A \) is reduced,

\[
c_i a_j = 0 \text{ for all } i = 0, 1, \ldots; j = 0, 1, 2, \ldots \text{ and } b_i (c_0 - 1) = 0 \text{ for all } i
\]

and \( b_i c_j = 0 \) for all \( j \geq 1 \). Hence \( \{c_0, c_1, \ldots \} \subseteq \text{ann}(a_0, a_1, \ldots) \) and \( b_i (c_0 - 1) = 0 \).

So \( c_0 \in \text{ann}(a_0, a_1, \ldots) \) and \( b_i c_0 = b_i \) for all \( i = 0, 1, \ldots \). Therefore the above condition holds.

Because any PF-ring is a PF-ring, every PF-ring is a reduced ring. On a reduced ring \( A \), a partial order relation can be defined by \( a \preceq b \) if \( ab = a^2 \). The following lemma is given in Brewer[3] and Brewer et al.[4].

LEMMA 3. The relation \( \preceq \) defined above on a reduced ring \( A \) is a partial order.

PROOF. Clearly the relation \( \preceq \) is reflexive. Now assume \( a \preceq b \) and \( b \preceq a \). Then \( ab = a^2 \) and \( ba = b^2 \). So, \((a-b)^2 = a^2 - 2ab + b^2 = 0 \). Because \( A \) is reduced \( a - b = 0 \),
or \( a = b \). To prove transitivity of \( \leq \), assume \( a \leq b \) and \( b \leq c \). So \( ab = a^2 \) and \( bc = b^2 \). Consider

\[
(ac - ab)^2 = a^2(c^2 - 2cb + b^2)
\]

\[
= a^2(c^2 - b^2)
\]

\[
= ab(c - b)(c + b)
\]

\[
= 0
\]

because \( b(c - b) = 0 \). Since \( A \) is reduced, \( ac - ab = 0 \) or \( ac = ab = a^2 \). Therefore \( a \leq b \).

**Theorem 4.** The power series ring \( A[[X]] \) is a PP-ring if and only if \( A \) is a PP-ring in which every increasing chain of idempotents of \( A \) with respect to \( \leq \) has a supremum which is an idempotent element in \( A \).

**Proof.** Assume \( A[[X]] \) is a PP-ring. Let \( a \in A \). Since \( A[[X]] \) is a PP-ring and idempotents in \( A[[X]] \) are in \( A \), \( \text{ann}(a) \in A[[X]] \). We claim \( \text{ann}(a) = eA[[X]] \). Because \( eA = 0 \), \( rea = 0 \) for all \( r \in A \). Hence \( eA \subseteq \text{ann}(a) \). Now let \( b \in \text{ann}(a) \). Hence

\[
b \in \text{ann}(a) \text{. Thus } b = eg(X) \text{ for some } g(X) = b_0 + b_1X + \ldots \text{. Consequently, } b = eb_0.\]

That is \( b \in eA \). Whence \( A \) is a PP-ring.

To complete the proof of this direction, let \( e_0 \leq e_1 \leq e_2 \ldots \) be an increasing chain of idempotents in \( A \). Because \( A[[X]] \) is a PP-ring and since idempotents of \( A[[X]] \) are in \( A \), \( \text{ann}(e_0 + e_1X + \ldots) = eA[[X]] \). Now we claim \( 1 - e = \sup\{e_0, e_1, \ldots\} \).

Since \( ee_i = 0 \), \( e_i(1 - e) = e_i, i = 0, 1, \ldots \).

So \( e_i \leq 1 - e \) for all \( i = 0, 1, \ldots \). Let \( y \) be an upper bound of \( \{e_0, e_1, \ldots\} \). So

\( e_i \leq y \) for \( i = 0, 1, \ldots \).

Hence \( 1 - y \in \text{ann}(e_0 + e_1X + \ldots) \).

Thus \( 1 - y = ec \) for some \( c \in A \). Consequently,

\[
y(1 - e) = (1 - ce)(1 - e)
\]

\[
= 1 - ec - e + ec
\]

\[
= 1 - e
\]

So \( 1 - e \leq y \). Therefore \( 1 - e = \sup\{e_0, e_1 \ldots\} \).

To prove the other way around, consider \( \text{ann}(f(X)) \) where \( f(X) = a_0 + a_1X + \ldots \).

Hence

\[
\text{ann}(f(X)) = \bigcap_{A[[X]]} \text{ann}(a_0, a_1, \ldots)\]

\[
\text{ann}(a_0, a_1, \ldots) = \bigcap_{i=0}^{\infty} \text{ann}(a_1)
\]
because $A$ is a PP-ring.

Let $d_0 = e_0$, $d_1 = e_0 e_1$, $\ldots$, $d_n = d_{n-1} e_n$, $\ldots$

One can easily check that

$$\bigcap_{i=0}^{\infty} e_i A = \bigcap_{i=0}^{\infty} d_i A$$

Also it is clear that

$$d_0 \geq d_1 \geq d_2 \ldots$$

Therefore

$$1 - d_0 \leq 1 - d_1 \leq 1 - d_2 \ldots$$

By assumption, this increasing chain of idempotents has a supremum which is an idempotent. Let

$$\text{Sup}\{1 - d_0, 1 - d_1, 1 - d_2, \ldots\} = d.$$ So

$$(1 - d_1) d = 1 - d_1$$

for all $i = 0, 1, \ldots$.

We claim that

$$\bigcap_{i=0}^{\infty} d_i A = (1 - d)A.$$  

Now $1 - d \geq d_1$. So $(1 - d)d_1 = 1 - d$. Hence

$$(1 - d)A \subseteq d_1 A \text{ for all } i = 0, 1, \ldots$$

Thus

$$(1 - d)A \subseteq \bigcap_{i=0}^{\infty} d_i A.$$  

Let $y \in \bigcap_{i=0}^{\infty} d_i A$. Then $y = d_i y_1$, $i, 0, 1, \ldots$.

Consequently

$$(1 - d_1)(1 - y) = 1 + d_1 y - d_1 - y = 1 - d_1$$

Because $yd_1 = d_1^2 = d_1^2 y_1 = d_1 y_1 = y$.

Therefore $1 - d_1 \leq 1 - y$ for all $i = 0, 1, \ldots$.
Because \[ d = \text{Sup}(1 - d_0, 1 - d_1, 1 - d_2, \ldots), \]
\[ d \neq 1 - y. \] So \[ d = d(1 - y) = d - dy \]

Hence \[ dy = 0. \] Thus \[ y(1 - d) = y - yd = y \]

That is \[ y \in (1 - d)A. \] Therefore \[ \bigcap_{i=0}^{\infty} d_i A = (1 - d)A. \]

Consequently,

\[
\text{ann } (f(X)) = (1 - d) \ A[[X]]
\]
\[
A[[X]]
\]

Therefore \[ A[[X]] \] is a PP-ring.

REFERENCES