AN INVERSE PROBLEM FOR HELMHOLTZ’S EQUATION

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ABSTRACT. The refraction coefficient in Helmholtz’s equation is found from the knowledge of a family of the solutions to this equation on two lines.

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1. INTRODUCTION.

Let
\[ [\nabla^2 + k^2 + k^2 v(x)] u = -\delta(x-y), \quad k > 0 \]
(1.1)
where \( x = (x_1, x_3), y = (y_1, y_3), v = v(x_1, x_3), u = u(x_1, x_3, y_1, y_3, k) \).

Assume that
\[ v(x) = 0 \text{ for } x_1 \geq a \text{ or } x_1 \leq -a, \text{ or } x_3 \geq 0 \text{ or } x_3 \leq -R, \quad v \in L^2 \]
(1.2)

Here \( R > 0 \) is an arbitrary large fixed number. Write (1.1) as
\[ u = g + k^2 \int g v g dz, \quad g := (i/4) H^0_l(k|x-y|) \]
(1.3)

where the integral is taken over the support of \( v \) and \( H^0_l \) is the Hankel function.

The problem is: find \( v(k) \) from the knowledge of \( u(-a, x_3, a, y_3, k) \) for all \( -\infty < x_3, y_3 < \infty \) and \( 0 < k < k_0 \), where \( k_0 > 0 \) is an arbitrary small number.

2. SOLUTION.

Let \( L = \{ x: x_1 = a, x_3 \in \mathbb{R} \} \), \( R^1 = (\infty, \infty) \). We use the method given in [1], [2]. It follows from (1.3) that
\[ f(x_3, y_3, k) := k^2 (u-g) = \int g v g dz + o(k) \text{ as } k \to 0, \quad x \in L_{-a}, \quad y \in L_a. \]
(2.1)

Let us take the Fourier transform of (2.1) in \( x_3 \) and \( y_3 \), define
\[ \hat{f}(\lambda, \mu) := (2\pi)^{-2} \int_{-\infty}^{\infty} \exp(-i\lambda x_3 - i\mu y_3) f(x_3, y_3) dx_3 dy_3, \text{ and use the formula } \]
\[ (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-i\lambda x_3) g(x, z) dx_3 = i(4\pi)^{-1} \exp(-i\lambda z_3^1) \exp\left\{ \frac{(k^2 - \lambda^2)^{1/2}}{(k^2 - \lambda^2)^{1/2}} \right\} \]
(2.2)
where \( x = (-\alpha, x_3) \), the radical \( (k^2 + 10 - \lambda^2)^{\frac{1}{2}} > 0 \) for \( \lambda^2 < k^2 \) and is defined by analytic continuation for all complex \( \lambda \) on the complex \( \lambda \)-plane with the cut \((-k, k)\), \( k > 0 \), so that
\[
(k^2 - \lambda^2)^{\frac{1}{2}} = i (\lambda^2 - k^2)^{\frac{1}{2}} \quad \text{if} \quad k^2 < \lambda^2. \tag{2.3}
\]

The result is
\[
\tilde{f}(\lambda, \mu) = \int dzv(z)h(\lambda, \mu, z, k) + o(k) \tag{2.4}
\]
where for \( k^2 > \lambda^2, k^2 > \mu^2 \), and \( r(\lambda) := (k^2 - \lambda^2)^{\frac{1}{2}} \) one has
\[
h := -(16\pi^2)^{-1}\exp(-i(\lambda + \mu)z_3 + i(a + z_1)r(\lambda) + i(a - z_1)r(\mu))r^{-1}(\lambda)r^{-1}(\mu) \tag{2.5}
\]
and for \( k^2 < \mu^2 \) and \( k^2 < \lambda^2 \) one uses (2.3).

In the Born approximation one drops the term \( o(k) \) in (2.4) and solves the resulting linear integral equation for \( v(z) \) [2].

In the exact theory one passed to the limit \( k \to 0 \) in (2.4), obtains a linear integral equation for \( v \) and solves this equation analytically [2]. It is not possible to pass to the limit \( k \to 0 \) in (2.1) because \( g(kr) = o(k) + g_0 + 0((kr)^2 \ln(k/2)) \) as \( k \to 0 \), where \( g_0 := (2\pi)^{-1}\ln(r^{-1}) \), \( \alpha(k) := -(2\pi)^{-1}\ln(k/2) + i\gamma/2\pi \), and \( \gamma = 0.5572 \ldots \) is Euler's constant. Thus \( g(kr) \) does not have a finite limit as \( k \to 0 \). Nevertheless one can pass to the limit \( k \to 0 \) in (2.4) if \( \gamma \neq 0 \) or \( \mu \neq 0 \). The reason is that the term \( \alpha(k) \) in (2.1) after the Fourier transform becomes \( \alpha(k)\delta(\lambda)\delta(\mu) \), and this term, which contains the factor \( \alpha(k) \to \infty \) as \( k \to 0 \), is zero for \( \lambda \neq 0 \) or \( \mu \neq 0 \). Another way to study the limit behavior of the solution to (2.1) is given in [2]. To give the exact theory, pass to the limit \( k \to 0 \) in (2.4) to get
\[
\int v(z) \exp(-ipz_3 + qz_1)dz_1dz_3 = \psi(p, q) \tag{2.6}
\]
where we used (2.5) and set
\[
p := \lambda + \mu, \quad q := |\mu| - |\lambda|, \tag{2.7}
\]
\[
\psi(p, q) := 16\pi^2\tilde{f}(\lambda, \mu)|\lambda| |\mu| \{\exp(|\lambda| + |\mu|)\} \tag{2.8}
\]
and the right side of (2.8) should be expressed as a function of \( (p, q) \) by formulas (2.7).

If \( \mu > 0 \) and \( \lambda > 0 \) then the point \((p, q)\) defined by (2.7) runs through \( Q_+ = \{p, q; \ |q| < p, p > 0\} \).

If \( \lambda < 0 \) and \( \mu < 0 \) then \( (p, q) \) runs through \( Q_- = \{p, q; \ |q| < -p, p < 0\} \). If \( \psi(p, q) \) is known in \( Q_+ \) or \( Q_- \) then \( v(z) \) can be uniquely recovered from (2.6) by the analytical methods given in [2] p. 331-341, where inversion of the Fourier and Laplace transforms of compactly supported functions from a compact set is given. This inversion problem is ill-posed and its numerical implementation is not a simple matter.

One can use the same ideas to solve equation (2.4) at a fixed \( k > 0 \) in the Born approximation. The basic equation analogous to (2.4) for the case when \(-k < \lambda, \mu < k\), is:
\[ f v(z) \exp(-i(pz_3+q_1z_1))dz = f(p,q_1) \text{ for } -k < \mu, \lambda < k \] (2.9)

where \( p = \lambda + \mu, \quad q_1 := r(\mu) - r(\lambda), \)

\[ F(p,q_1) := -16\pi^2 \frac{\xi(\lambda,\mu)r(\lambda)r(\mu)}{a[r(\lambda)+r(\mu)]} \] (2.10)

and the right side of (2.10) should be expressed as a function of \( p, q_1 \).

If \((\lambda,\mu) \in (\lambda_1,\lambda_2) \times (\mu_1,\mu_2) \) are real then the basic equation in the Born approximation is equation (2.6) in which the right side is now given by the formula \( \psi = F \), where \( F \) is defined by (2.10) and in (2.10) the radicals \( r(\lambda) \) and \( r(\mu) \) are computed by formula (2.3) for \( \lambda^2 > k^2 \) and \( \mu^2 > k^2 \).

Equation (2.9) can also be solved analytically with the prescribed accuracy by the methods given in [2].

The problem considered is of interest in application.

REFERENCES

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