UNSTABLE PERIODIC WAVE SOLUTIONS OF
NERVE AXION DIFFUSION EQUATIONS

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ABSTRACT. Unstable periodic solutions of systems of parabolic equations are
studied. Special attention is given to the existence and stability of solutions.

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stability analysis

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1. INTRODUCTION.

Diffusion systems of partial differential equations are of great importance in
biosciences. In this paper, unstable periodic solutions of systems of the form

\[ \begin{align*}
\frac{\partial u}{\partial t} &= u_{xx} + F(u, w), \\
\frac{\partial w}{\partial t} &= G(u, w),
\end{align*} \tag{1.1} \]

are studied. Equations of this type arise in neurophysiology in the study of nerve
impulses on nerve axon, see [1,2]. Other classes of diffusion equations are also
involved in biology, see for example [3-9].

2. EXISTENCE OF SOLUTIONS

It is known that for \( G(u, w) = \varepsilon u, \) if \( \varepsilon > 0 \) is sufficiently small, equation
(1.1) has two types of wave solutions, namely, pulse travelling wave solutions and
periodic travelling wave solutions. A travelling wave solution is a solution of
equation (1.1) of the form

\[ [u(x, t), w(x, t)] = [\phi(z; c), \psi(x; c)], \ z = x + ct, \]

hence \([\phi(z; c), \psi(z; c)]\) satisfies the ordinary differential equation

\[ \begin{align*}
\frac{d^2 \phi}{dz^2} - c \frac{d \phi}{dz} + F(\phi, \psi) &= 0, \\
- c \frac{d \psi}{dz} + G(\phi, \psi) &= 0.
\end{align*} \tag{2.1} \]

A pulse travelling wave solution is a non-constant solution of (2.1) satisfying
and a periodic travelling wave solution is a periodic solution of (2.1).

In [10], Evans showed that equation (1.1) has two pulse travelling solutions with different propagation speeds \(c_1\) and \(c_2\). On the existence of periodic travelling wave solutions, Hastings [11] showed that equation (2.1) with \(G(u,w) = cu\) has a non-constant periodic solution if \(c > 0\) is sufficiently small and the speed \(c\) is limited to a certain range. Rinzel and Keller [12] studied the case in which \(F(u,w)\) is a function of \(u\) only given by

\[
F(u,w) = \begin{cases} 
  u & \text{for } u \leq a, \\
  u-1 & \text{for } a < u,
\end{cases}
\]

where \(0 < a < \frac{1}{2}\). Under this assumption, equation (2.1) has a non-constant periodic solution if \(c\) is limited in the range \(c_1 < c < c_2\) and the period \(p(c)\) is a smooth function of \(c\). They demonstrated the behavior of the function \(p(c)\) under the two cases when \(a\) is not very small and when \(a\) is very small. Dai [13] proved the existence and uniqueness of solutions for a general case and studied stability of the solution.

### 3. STABILITY ANALYSIS.

Stability of periodic travelling wave solutions is related to the eigenvalues of a matrix in the following theorem. Let \(A(z;\lambda, c)\) be the matrix

\[
A(z;\lambda, c) = \begin{bmatrix} 
  0 & 1 & 0 \\
  \lambda - F_1[\psi(z; c), \psi(z; c)] & c & -F_2[\psi(z; c), \psi(z; c)] \\
  G_1[\psi(z; c), \psi(z; c)] & 0 & G_2[\psi(z; c), \psi(z; c)] - \lambda \\
\end{bmatrix}
\]

where \(F_1\) and \(G_1\) denote the partial derivatives as usual, and let \(X(z;\lambda, c)\) be a matrix satisfying the differential equation

\[
\frac{d}{dz} X = A X
\]

with the initial condition \(X(0;\lambda, c) = I\).

**THEOREM 3.1.** Suppose the functions \(F\) and \(G\) in equation (1.1) satisfy (a) \(F(0,0) = 0\), (b) \(G(0,0) = 0\) and (c) the matrix \(X(p(c);\lambda, c)\) has an eigenvalue of modulus 1, for some complex number \(\lambda\) with \(\text{Re} \lambda > 0\), then a periodic travelling wave solution \([\psi(z; c), \psi(z; c)]\) is unstable.

**PROOF.** With the change of variables,

\[
z = x + ct, \\
t = t, \\
[u(x,t), w(x,t)] = [\bar{u}(z,t), \bar{w}(z,t)],
\]

and
equation (1.1) becomes

\[ \begin{align*}
\tilde{u}_t &= \tilde{u}_{zz} - c \tilde{u}_z + F(\tilde{u},\tilde{w}), \\
\tilde{w}_t &= -c \tilde{w}_z + G(\tilde{u},\tilde{w}).
\end{align*} \tag{3.1} \]

The linearized perturbation equation of the above system with respect to the solution \([\phi(z;c), \psi(z;c)]\) is

\[ \begin{align*}
\ddot{U}_t &= \ddot{U}_{zz} - c \dot{U}_z + F_1[\phi,\psi] \dot{U} + F_2[\phi,\psi] \dot{W}, \\
\ddot{W}_t &= -c \ddot{W}_z + G_1[\phi,\psi] \dot{U} + G_2[\phi,\psi] \dot{W},
\end{align*} \tag{3.2} \]

where \(\phi = \phi(z;c)\) and \(\psi = \psi(z;c)\), since \(F(0,0) = G(0,0) = 0\). Equation (3.2) has a solution of the form

\[ \begin{align*}
\tilde{U}(z,t) &= e^{\lambda t} y_1(z;\lambda), \\
\tilde{W}(z,t) &= e^{\lambda t} y_2(z;\lambda),
\end{align*} \]

where \((y_1, y_2)\) satisfies the following system of linear ordinary differential equations

\[ \begin{align*}
\lambda y_1 &= \frac{d^2 y_1}{dz^2} - c \frac{dy_1}{dz} + F_1[\phi,\psi] y_1 + F_2[\phi,\psi] y_2, \\
\lambda y_2 &= -c \frac{dy_2}{dz} + G_1[\phi,\psi] y_1 + G_2[\phi,\psi] y_2,
\end{align*} \tag{3.3} \]

where \(\phi = \phi(z;c)\) and \(\psi = \psi(z;c)\). Note that if equation (3.3) has a solution which is bounded for all \(z\) in \((-\infty, \infty)\) for a number \(\lambda\) with \(\text{Re}(\lambda) > 0\), then equation (3.2) has a solution \([\tilde{U}(z,t), \tilde{W}(z,t)]\) which grows exponentially, and hence, the travelling wave solution \([\phi(z;c), \psi(z;c)]\) is unstable.

Using Floquet's theory, we can show that equation (3.3) has a bounded non-trivial solution if and only if one of the eigenvalues of \(X(p(c);\lambda,c)\) is a modulus 1. Equation (3.3) can be rewritten as

\[ \begin{align*}
\frac{d}{dz} \left( \frac{dy_1}{dz} \right) &= \left( \lambda - F_1[\phi,\psi] \right) y_1 + c \frac{dy_1}{dz} - F_2[\phi,\psi] y_2, \\
\frac{c}{dz} \frac{dy_2}{dz} &= G_1[\phi,\psi] y_1 + (G_2[\phi,\psi] - \lambda) y_2,
\end{align*} \]

and so can be represented by the matrix differential equation

\[ \frac{d}{dz} \mathbf{v} = A(z;\lambda,c) \mathbf{v}, \]
where
\[ y = \begin{bmatrix} y_1 \\ \frac{dy_1}{dz} \\ y_2 \end{bmatrix} \]
and the matrix \( A \) is as defined before. Now, since the coefficient matrix \( A(z;\lambda,c) \) is a \( p(c) \)-periodic function of \( z \), Floquet's theory yields that equation (3.3) has a bounded non-trivial solution if and only if one of the eigenvalues of the matrix \( X(p(c);\lambda,c) \) defined before is of modulus 1. The proof is now complete.

In the following lemma, it is shown that under the special case \( \lambda = 0 \), one eigenvalue of \( X(p(c);0,c) \) is unity and the product of the other two eigenvalues is greater than one.

**LEMMA 3.1.** Suppose (a) \( G_2(u,w) \geq 0 \) for all \( u \) and \( w \) and (b) \( \lambda = 0 \), let \( \mu_i(\lambda,c), i = 1, 2, 3, \) denote the eigenvalues of \( X(p(c);\lambda,c) \), then one eigenvalue, say
\[ \mu_1(0,c) = 1, \]
and
\[ \mu_2(0,c) \mu_3(0,c) > 1. \]

**PROOF.** Differentiation of equation (2.1) leads to
\[ \frac{d}{dz} \left( \frac{d^2 \phi}{dz^2} \right) = c \frac{d}{dz} \left( \frac{d\phi}{dz} \right) - F_1[\phi,\psi] \frac{d\phi}{dz} - F_2[\phi,\psi] \frac{d\psi}{dz}, \]
where \( \phi = \phi(z;c) \) and \( \psi = \psi(z;c) \). Therefore the vector
\[ v = \begin{bmatrix} \phi \\ \frac{d\phi}{dz} \\ \psi \end{bmatrix} \]
satisfies the matrix equation
\[ \frac{d}{dz} w = A(z;0,c) w, \]
that is,
\[ \begin{bmatrix} \frac{d\phi}{dz} \\ \frac{d^2 \phi}{dz^2} \\ \frac{d\psi}{dz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -F_1[\phi(z;c),\psi(z;c)] & c & -F_2[\phi(z;c),\psi(z;c)] \\ G_1[\phi(z;c),\psi(z;c)] & 0 & G_2[\phi(z;c),\psi(z;c)] \end{bmatrix} \begin{bmatrix} \frac{d\phi}{dz} \\ \frac{d^2 \phi}{dz^2} \\ \frac{d\psi}{dz} \end{bmatrix}. \]
We know that (see for example, Sanchez)

\[
\frac{\partial w(z,c)}{\partial z} = X(z;0,c) w(z;0,c)
\]

and since \(w(z,c)\) is a \(p(c)\) - periodic function of \(z\), it follows that

\[
\frac{\partial w(z;0,c)}{\partial z} = \frac{\partial w(p(c);c)}{\partial z} = X(p(c);0,c) \frac{\partial w(z;0,c)}{\partial z}.
\]  

(3.5)

Thus there is an eigenvalue, say

\[
\mu_1(0,c) = 1.
\]

Further, by Jacobi's formula,

\[
\det \{X(z;\lambda,c)\} = \det X(0;\lambda,c) \exp \int_0^z \text{tr} \{A(\xi;\lambda,c)\} \, d\xi
\]

\[
= (1) \exp \int_0^z \left( c + \frac{G_2[\phi,\psi]}{c} \right) \, d\xi.
\]

In particular,

\[
\det \{X(p(c);0,c)\} = \exp [c p(c)] \exp \int_0^{p(c)} \frac{G_2[\phi,\psi]}{c} \, d\xi
\]

\[
> 1
\]

since \(c > 0\), \(p(c) > 0\) and \(G_2(u,w) \geq 0\) for all \(u,w\).

But \(\det \{X(p(c);0,c)\} = \mu_1(0,c) \mu_2(0,c) \mu_3(0,c)\) and

\[
\mu_1(0,c) = 1, \text{ hence } \mu_2(0,c) \mu_3(0,c) > 1.
\]

Note that under the assumptions of Lemma 3.1, either \(|\mu_2(\lambda,c)| > 1\) or \(|\mu_3(\lambda,c)| > 1\) for \(\lambda\) sufficiently small. In the next theorem, we will see that if \(L(c)\) is decreasing, i.e. \(L'(c) < 0\), then \(\mu_1(\lambda,c)\) is increasing at \(\lambda = 0\), i.e.

\[
\frac{\partial}{\partial \lambda} \mu_1(\lambda,c) \bigg|_{\lambda=0} > 0.
\]

**THEOREM 3.2.** Suppose (a) \(p'(c) < 0\), then \(\frac{\partial}{\partial \lambda} \mu_1(\lambda,c) \bigg|_{\lambda=0} > 0\), and hence if (b) the assumptions in Lemma 3.1 also hold, then \(\mu_1(\lambda,c) > 1\) for \(\lambda\) sufficiently small.

**PROOF:** We claim that the following equality

\[
\frac{\partial}{\partial \lambda} \mu_1(\lambda,c) \bigg|_{\lambda=0} = -p'(c)
\]

actually holds.
Recall the vector $w(z;c)$, namely,

$$w = \begin{bmatrix} \phi \\ \frac{d\phi}{dz} \\ \psi \end{bmatrix}$$

which satisfies the periodicity

$$w(p(c);c) = w(0;c).$$

Differentiation of the above equation with respect to $c$ leads to

$$w_z(p(c);c) p'(c) + w_c(p(c);c) = w_c(0;c). \quad (3.6)$$

Let $v = [v_1^*(z;\lambda,c), v_2^*(z;\lambda,c)]$ be a solution of equation (3.3) satisfying the initial condition

$$v(0;\lambda,c) = w_c(0;c) + \lambda w_c(0;c), \quad (3.7)$$

where $v(z;\lambda,c)$ is the vector defined before. We have observed before that $[\frac{d\phi}{dz}(z;c), \frac{d\psi}{dz}(z;c)]$, which satisfies equation (3.4), is a solution of equation (3.3) under $\lambda=0$.

In view of the condition (3.7) and by uniqueness of solutions, we have

$$v^*(z;0,c) = w_{z}(z;c). \quad (3.8)$$

Differentiation of equation (3.3) with respect to $\lambda$ leads to

$$y_1 + \lambda \frac{\partial y_1}{\partial \lambda} = \frac{d^2}{dz^2} \left( \frac{\partial y_1}{\partial \lambda} \right) - c \frac{d}{dz} \left( \frac{\partial y_1}{\partial \lambda} \right) + F_1[\phi,\psi] \frac{\partial y_1}{\partial \lambda} + F_2[\phi,\psi] \frac{\partial y_2}{\partial \lambda},$$

$$y_2 + \lambda \frac{\partial y_2}{\partial \lambda} = -c \frac{d}{dz} \left( \frac{\partial y_2}{\partial \lambda} \right) + G_1[\phi,\psi] \frac{\partial y_1}{\partial \lambda} + G_2[\phi,\psi] \frac{\partial y_2}{\partial \lambda}. \quad (3.9)$$

Under $\lambda = 0$, and replacing $[y_1, y_2]$ by $[y_1^*, y_2^*]$, equation (3.9) by equality (3.8) becomes

$$\frac{d\phi}{dz}(z;c) = \frac{d^2}{dz^2} \left( \frac{\partial y_1^*}{\partial \lambda} \right) - c \frac{d}{dz} \left( \frac{\partial y_1^*}{\partial \lambda} \right) + F_1[\phi,\psi] \frac{\partial y_1^*}{\partial \lambda} + F_2[\phi,\psi] \frac{\partial y_2^*}{\partial \lambda},$$

$$\frac{d\psi}{dz}(z;c) = -c \frac{d}{dz} \left( \frac{\partial y_2^*}{\partial \lambda} \right) + G_1[\phi,\psi] \frac{\partial y_1^*}{\partial \lambda} + G_2[\phi,\psi] \frac{\partial y_2^*}{\partial \lambda}. \quad (3.10)$$

where $\frac{\partial y_1^*}{\partial \lambda} = \frac{\partial y_1^*}{\partial \lambda}(z;0,c)$ now. On the other hand, differentiating equation (2.1) with respect to $c$, we get
\[
\frac{d^2}{dz^2} \left( \frac{\partial \phi}{\partial c} \right) - c \frac{d}{dz} \left( \frac{\partial \phi}{\partial c} \right) + F_1(\phi, \psi) \frac{\partial \phi}{\partial c} + F_2(\phi, \psi) \frac{\partial \psi}{\partial c} = 0,
\]

\[
- \frac{d}{dz} \left( c \frac{\partial \psi}{\partial c} \right) + G_1(\phi, \psi) \frac{\partial \phi}{\partial c} + G_2(\phi, \psi) \frac{\partial \psi}{\partial c} = 0, \tag{3.11}
\]

where \( \phi = \phi(z; c) \) and \( \psi = \psi(z; c) \). Therefore both \( \frac{\partial \psi}{\partial \lambda}(z; 0; c), \frac{\partial \phi}{\partial \lambda}(z; 0; c) \) and \( \frac{\partial \phi}{\partial c}(z; c), \frac{\partial \psi}{\partial c}(z; c) \) satisfy the same differential equation. In addition, differentiation of the initial condition (3.7) yields

\[
v_\lambda(0; \lambda, c) = \omega_c(0; c),
\]

in particular,

\[
v_\lambda(0; 0, c) = \omega_c(0; c)
\]

and hence the equality

\[
v^*_\lambda(z; 0, c) = \omega_c(z; c), \quad 0 \leq z \leq p(c). \tag{3.12}
\]

The equalities (3.8) and (3.12) together give

\[
v^*(z; \lambda, c) = \omega_c(z; c) + \lambda \omega_c(z; c) + O(\lambda^2), \tag{3.13}
\]

\[0 \leq z \leq p(c), \text{ as } \lambda \to 0.
\]

Knowing \( v^*(z; \lambda, c) = X(z; \lambda, c) v^*(0; \lambda, c) \), by equation (3.13) for \( z = p(c) \) and also \( z = 0 \), we get

\[
\omega_c(p(c); c) + \lambda \omega_c(p(c); c) + O(\lambda^2)
\]

\[= X(p(c); \lambda, c) [\omega_c(0; c) + \lambda \omega_c(0; c)]. \tag{3.14}
\]

Substitution of the equation (3.6) containing \( p'(c) \) into the left hand side of equation (3.14) and periodicity lead to

\[
X(p(c); \lambda, c) [\omega_c(0; c) + \lambda \omega_c(0; c)]
\]

\[= [1 - \lambda p'(c)] [\omega_c(0; c) + \lambda \omega_c(0; c)] + O(\lambda^2).
\]

Hence the eigenvalue \( \mu_1(\lambda, c) \) satisfies

\[
\frac{\partial}{\partial \lambda} \mu_1(\lambda, c) \bigg|_{\lambda=0} = -p'(c).
\]

The proof is now complete.
On the other hand, under certain conditions, two eigenvalues have modulus less than one and one has modulus greater than one.

**Theorem 3.3.** Suppose (a) $F_2(u,w)$ is a non-zero constant and (b) $G_1(u,w)$ and $G_2(u,w)$ are constant, then for $\lambda$ sufficiently large, two eigenvalues of $X(p(c); \lambda, c)$ have modulus $<1$ and one has modulus $>1$.

**Proof:** Decompose the matrix $A(z; \lambda, c)$ as follows

$$A(z; \lambda, c) = B(\lambda, c) + E(z; c)$$

Let $s_i(\lambda, c)$, $i = 1, 2, 3$ be the eigenvalues of $B(\lambda, c)$ and $q_i$ the corresponding eigenvectors. The characteristic equation of $B(\lambda, c)$ is

$$-s^3 + \left(\frac{G_2 - \lambda}{c} + c\right)s^2 + (2\lambda - G_2)s + \lambda \left(\frac{G_2}{c} - \frac{F_2 G_1}{c}\right) = 0.$$
Now consider the matrix

\[ Y(z; \lambda, c) = Q^{-1} X(z; \lambda, c) Q \]

which has the same eigenvalues as \( X(z; \lambda, c) \), in particular with \( z = p(c) \), and satisfies the differential equation

\[
\frac{d}{dz} Y(z; \lambda, c) = Q^{-1} A(z; \lambda, c) Q Y(z; \lambda, c) \\
= [Q^{-1} B(\lambda, c) Q + Q^{-1} E(z; c) Q] Y(z; \lambda, c),
\]

since \( \frac{d}{dz} X(z; \lambda, c) = A(z; \lambda, c) X(z; \lambda, c) \).

But \( Q^{-1}EQ \) is the diagonal matrix from before and it can be shown easily using (3.15) and (3.16) that all elements of \( Q^{-1}EQ \) are \( o(1) \) as \( \lambda \to \infty \), therefore the eigenvalues of \( Y(p(c); \lambda, c) \) and hence of \( X(p(c); \lambda, c) \) approach

\[ \exp\left[ s_i(\lambda, c) p(c) \right], \ i = 1, 2, 3 \ \text{as} \ \lambda \to \infty. \]

It follows from (3.15) that as \( \lambda \to \infty \), two eigenvalues of \( X(p(c); \lambda, c) \) have modulus \( < 1 \) and one has modulus \( > 1 \).

To summarize, under the assumptions of both Theorem (3.2) and Theorem (3.3), at least two eigenvalues of \( X(p(c); \lambda, c) \) have modulus \( > 1 \) as \( \lambda \to 0^+ \), and two eigenvalues of \( X(p(c); \lambda, c) \) have modulus \( < 1 \) as \( \lambda \to \infty \). Hence one of the eigenvalues must have modulus \( \approx 1 \) for some \( \lambda > 0 \) and under Theorem (3.1), the travelling wave solution \( (\phi(z; c), \psi(z; c)) \) is unstable.

REFERENCES


