ON SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT: Let $m_1, m_2$ be any numbers and let $V_{m_1, m_2}$ be the class of functions of analytic in the unit disc $E=\{z: |z|<1\}$ for which

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}}$$

where $S_1$ and $S_2$ are analytic in $E$ with $S_1'(0)=S_2'(0)=1$. Moulis [1] gave a sufficient condition and a necessary condition on parameters $m_1$ and $m_2$ for the class $V_{m_1, m_2}$ to consist of univalent functions if $S_1$ and $S_2$ are taken to be convex univalent functions in $E$. In fact he proved that if $f \in V_{m_1, m_2}$ where $S_1$ and $S_2$ are convex and

$$m_1 = \frac{k+2}{4} e^{-i\alpha} (1-\rho) \cos \alpha, \quad m_2 = \frac{k-2}{4} (1-\rho) e^{-i\alpha} \cos \alpha, \quad 2|m_1 + m_2| \leq 1,$$

then $f$ is univalent in $E$.

In this paper we consider the class $V_{m_1, m_2}$ in more general way and show that it contains the class of functions with bounded boundary rotation and many other classes related with it. Some coefficient results, arclength problem, radius of convexity and other problems are proved for certain cases. Our results generalize many previously known ones.

KEY WORDS AND PHRASES. Univalent functions, boundary rotation, radius of convexity.

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1. INTRODUCTION.

Let $V_k^\alpha(\rho)$ be the class of all functions $f$, analytic in $E=\{z: |z|<1\}$, $f'(0)=1, f(0)=0, f'(z)\neq 0$ such that for $z=\rho e^{i\theta}, 0<\rho<1$

$$\int_0^{2\pi} \left| \frac{\Re \left( zf'(z) \right)' - \rho \cos \alpha \rho \cos \alpha}{1-\rho} \right| d\theta < k\pi \cos \alpha,$$

where $k \geq 2, 0<\rho<1, \alpha$ real and $|\alpha| < \frac{\pi}{2}$. 
The class \( V_k^\alpha(\rho) \) has been introduced and studied by Moulis in [1]. For \( \rho=0 \), we obtain the class \( V_k^\alpha \) introduced and studied in [2]. \( \rho=0 \) and \( \alpha=0 \) give us the well known class \( V_k \) of functions with bounded boundary rotation first introduced and discussed by Paatero [3] and Lowner [4]. Functions in \( V_k^\alpha \) and \( V_k^\alpha(\rho) \) may not possess boundary rotation.

Also a class \( T_k^\alpha(\rho) \) of analytic functions which is a generalization of \( V_k^\alpha(\rho) \) has been discussed in [5]. A function \( f \), analytic in \( E, f(0)=0=f'(0)-1 \) is in \( T_k^\alpha(\rho) \) if for \( z \in E \), there exists a function \( g \) in \( V_k^\alpha(\rho) \) such that

\[
\Re \frac{f'(z)}{g'(z)} > 0
\]

The cases when \( \rho=0 \) and \( \rho=0, \alpha=0 \) have been discussed in [6] and [7] respectively.

**Definition 1.1**

Let \( m_1 \) and \( m_2 \) be any numbers and \( S_1 \) and \( S_2 \) be analytic functions in \( E \) with \( S_1(0)=0=S_2(0) \) and \( S_1'(0)=1=S_2'(0) \). Then \( f \in V_{m_1,m_2} \) if and only if

\[
f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}}
\]

We have the following special cases.

**Case A.** Let \( m_1 = \frac{k+2}{4}, m_2 = \frac{k-2}{4}, k>2 \) in (1.1). Then

(i) \( V_{m_1,m_2} = V_k(0) \), the class of functions with bounded boundary rotation if \( S_1 \) and \( S_2 \) are convex univalent functions. This was proved by Brannan in [8].

(ii) \( V_{m_1,m_2} \equiv T_k^0(0) = T_k(0) \) if \( S_1 \) and \( S_2 \) are close-to-convex univalent functions, see [7].

(iii) \( V_{m_1,m_2} \) coincides with \( V_k^\alpha \) if \( zS_1' \) and \( zS_2' \) are \( \alpha \)-spirallike functions. This result is shown in [2].

(iv) \( V_{m_1,m_2} \equiv T_k^\alpha \) if \( S_1 \) and \( S_2 \in T_k^\alpha(0) \), see [6] and \( V_{m_1,m_2} \equiv T_k^\alpha(\rho) \) if \( S_1 \) and \( S_2 \in T_k^\alpha(\rho) \), see [5].

**Case B.** Let \( S_1 \) and \( S_2 \) be convex univalent functions in (1.1). Then we have the following subcases:

(i) If \( m_1 = \frac{k+2}{4} e^{-i\alpha} \cos \alpha, m_2 = \frac{k-2}{4} e^{-i\alpha} \cos \alpha \), then \( f \in V_k^\alpha \) in (1.1). See [2].

(ii) If \( m_1 = \frac{k+2}{4}(1-\rho) e^{-i\alpha} \cos \alpha, m_2 = \frac{k-2}{4}(1-\rho) e^{-i\alpha} \cos \alpha \), then \( f \in V_k^\alpha(\rho) \) in relation (1.1). This is shown in [1].

2. **MAIN RESULTS**

We now proceed to prove the main results for the class \( V_{m_1,m_2} \). Wherever needed, certain restrictions on the parameters \( m_1 \) and \( m_2 \) for analytic functions \( S_1 \) and \( S_2 \) will be imposed.

**Theorem 2.1**

Let \( f \in V_{m_1,m_2} \) such that
where $S_1$ and $S_2$ are convex univalent in $E$. Let

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}}$$

where $0 < r < 1$ and $2m_1 \lambda > 1$, $m_1, m_2 > 0$

Then

$$\limsup_{r \to 1} \frac{2m_1 \lambda - 1}{(1-r)} I_\lambda (r) \leq A(m_1, m_2, \lambda)$$

where

$$A(m_1, m_2, \lambda) = \frac{2m_2 \lambda}{\pi^\lambda (2m_1 \lambda - 1) \Gamma(m_1 \lambda)}$$

Proof

$$I_\lambda (r) = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{S_1'(z)}{S_2'(z)} \right|^\lambda d\theta$$

Then $|S_2'(z)| \geq \frac{1}{2}$ by the distortion theorems for convex functions [9] and $S_1'$ is subordinate to $(1-z)^{-2}$ in $E$. Consequently

$$I_\lambda (r) \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{S_1'(z)}{S_2'(z)} \right|^\lambda d\theta = (1+r)^{2m_2 \lambda} J_{2m_1 \lambda} (r)$$

Now it has been shown by Pommerenke in [10] that

$$J_\lambda (r) = \frac{\Gamma(p-1)}{2^{p-1} \Gamma^2(p)} \frac{1}{(1-r)^p-1}, \quad r > 1$$

Using the recurrence and duplication formulae for the Gamma function.

Substitution of (2.3) in (2.2) completes the proof.

Corollary 2.1

Let $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$. Then $f \in V_k$

and

$$\limsup_{r \to 1} \frac{1}{2} (\frac{k-1}{2}) \lambda - 1 I_\lambda (r) \leq A(k, \lambda),$$

where

$$A(k, \lambda) = \frac{\Gamma(\frac{1}{2} k \lambda - \frac{1}{2})}{\pi^\lambda (\frac{1}{2} k \lambda - 1) \Gamma(\frac{1}{4} k \lambda + \frac{1}{2})}$$
This result was proved in [8].

**Theorem 2.2**

Let $f \in V_{m_1, m_2}$ and $S_1, S_2$ be convex functions. Let $L(r)$ denote the length of the arc $f(r) = r$ given by the formula for $z = re^{i\theta}$.

$$L(r) = \int_0^{2\pi} |zf'(z)| d\theta$$

Then, for $m_1 > \frac{1}{2}$, $m_2 > 0$, we have

$$L(r) = \frac{1}{(1-r)^{2m_1-1}}$$

where $0(1)$ is a constant depending only on $m_1$ and $m_2$.

The proof follows immediately from Theorem 2.1 by taking $\lambda = 1$.

From Theorem 2.1 and the standard inequality [9, p. 11],

$$|a_n| \leq \frac{\pi}{n} (1- \frac{1}{2})$$

we have the following.

**Theorem 2.3**

Let $f \in V_{m_1, m_2}$ and be given by (1.1) with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ where $S_1$ and $S_2$ in (2.1) are convex, $m_1 > \frac{1}{2}, m_2 > 0$. Then for $n \geq 2$

$$\lim_{n \to \infty} \sup n^{2-2m_1} |a_n| \leq \frac{e^{2m_2} \Gamma(m_1 + \frac{1}{2})}{(2m_1-1) \Gamma(m_1)}$$

**Corollary 2.2**

If $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$ in Theorem 2.3 then $f \in V_k$ and

$$\lim_{n \to \infty} \sup \frac{1}{n^{\frac{1-k}{2}}} |a_n| \leq \frac{e^{\frac{1}{2}k} \Gamma(k+1)}{\pi^k (k) \Gamma(k+\frac{1}{2})}$$

This result was proved in [8].

**Theorem 2.4**

Let $f \in V_{m_1, m_2}$ with $S_1$ and $S_2$ convex and $m_1 > 1$, $m_2 > 0$. Let $f$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then for $n \geq 1$

$$|a_{n+1}| - |a_n| \leq C(m_1, m_2)^{2m_1-3} n$$

where $C(m_1, m_2)$ is a constant depending only on $m_1$ and $m_2$.

**Proof**

For $z_1 \in \mathbb{E}$ and $n \geq 1$, we have

$$|(n+1)z_1 a_{n+1} - na_n| = \frac{1}{2\pi n^{m_1+1}} \int_0^{2\pi} |z-z_1| |zf'(z)| d\theta, \quad z = re^{i\theta}$$

$$= \frac{1}{2\pi n^3} \int_0^{2\pi} |z-z_1| |S_1'(z)|^{m_1} |S_2'(z)|^{m_2} d\theta$$

(2.4)
It is known [9] that for convex univalent functions $S_2$
\[ |S'_2(z)| \geq \frac{1}{(1+r)^2} \tag{2.5} \]

Also, by a result of Golusin [11], there exists a $z_1 \in \mathbb{E}$ with $|z_1| = r$
such that for all $z$, $|z| = r$
\[ |z-z_1| |S_1^*(z)| \leq \frac{2r^2}{1-r^2}, \tag{2.6} \]

where $S_1^*(z) = zS_1'(z)$ is univalent

Using (2.5) and (2.6), (2.4) becomes
\[ 2m \frac{(n+1)a_{n+1} - na_n}{2\pi r^{n-1}} \left( \frac{2r^2}{1-r^2} \right) \leq \frac{2m_2-1}{\pi r^{n-3}} \cdot \frac{1}{2m_1-2} \]

where we have used subordination for the function $S_1'$.

Putting $|z_1| = r$, $r = \frac{n}{n+1}$, we obtain the required result.

**Corollary 2.3**

Taking $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$, $k \geq 2$, we obtain $f \in V_k$ and $\|a_n\| \leq C(k) n^{\frac{k-2}{2}}$, where $C(k)$ is a constant depending only on $k$.

Now we give the radius of convexity problem for the class $V_{m_1, m_2}$

where the functions $S_1$ and $S_2$ are in $V_k$.

**Theorem 2.5**

Let $f \in V_{m_1, m_2}$ such that
\[ f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}}, \]

where $S_1, S_2 \in V_k$ and $m_1, m_2 > 0$ and real. Then $f$ is convex for $|z| < r$ where
\[ r = \frac{1+m_2(1-\frac{k}{2})}{k(m_1+m_2)+2m_1-m_2(1+\frac{k}{2})-1} \tag{2.7} \]

**Proof**

From definition it easily follows that
\[ \frac{(zf'(z))'}{f'(z)} = m_1 \frac{(zS_1'(z))'}{S_1'(z)} - m_2 \frac{(zS_2'(z))'}{S_2'(z)} + (1-m_1 + m_2) \]

Now, for $S_1 \in V_k$ it is known [12] that
\[ \text{Re} \left( \frac{(zS_1'(z))'}{S_1'(z)} \right) \geq \frac{1-kr+r^2}{1-r^2} \tag{2.8} \]
Also, by the Paatero representation theorem [3] we have, for $S_2 \in V_k$,

\[
\frac{(zS_2'(z))'}{S_2'(z)} = \frac{k+2}{4} h_1(z) - \frac{k-2}{4} h_2(z), \text{ Re } h_1(z) > 0, \quad i=1,2, \text{ and } h_1(0) = 1
\]

so that

\[
\text{Re} \left( \frac{(zS_2'(z))'}{S_2'(z)} \right) \leq \left| \frac{(zS_2'(z))'}{S_2'(z)} \right| \leq \frac{k}{2} \frac{1+r}{1-r}
\]

Thus, using (2.8) and (2.9), we have

\[
\text{Re} \left( \frac{zf'(z)}{f'(z)} \right) < \frac{[1+m_2(1-\frac{k}{2}) - k(m_1+m_2)r + [2m_1-m_2(1+\frac{k}{2})-1]r^2}{1-r^2}
\]

and this gives us the required result.

**Corollary 2.4**

If $k=2$, then $S_1, S_2 \in V_2 = \mathbb{C}$, the class of convex functions and equation (2.7) reduces to

\[
1-2(m_1+m_2)r+(2m_1-2m_2-1)r^2=0
\]

and in this case if $m_1 = \frac{k+2}{4}, m_2 = \frac{k-2}{4}$ then $V_{m_1, m_2}$ reduces to $V_k$ and equation (2.7) reduces to the known result

\[
1-kr+r^2 = 0
\]

which was given in [12].

**Corollary 2.5**

If $m_1 > 0, m_2=0$, then $f$ is convex for $|z| < r$, where $r$ is the least positive root of

\[
1-kr+(2a-1)r^2 = 0
\]

This result has been proved in [13].

**Theorem 2.6**

Let $f \in V_{m_1, m_2}$ such that

\[
f'(z) = \frac{(zS_1'(z))^{m_1}}{(zS_2'(z))^{m_2}}
\]

and $S_1, S_2 \in V_k, m_1, m_2 > 0, m_1-m_2 \leq 1$.

Then $f \in V_{k'},$ where $k' = \{ m_1(k-2)+m_2(k+2)+2 \}$

From the above result, we deduce the following:

(i) If $S_1, S_2 \in V_2$, then $f \in V_{4m_2+2}$ and in this case if $m_1 = \frac{k+2}{4}, m_2 = \frac{k-2}{4}$, we have the well known result [8] that $f \in V_k$.

(ii) If $m_1 = \alpha, m_2=0, 0 \leq \alpha \leq 1$, then $f \in V_{\alpha(k-2)+2}$. 
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