ON M-IDEALS IN $B(\bigoplus_{i=1}^{\infty} \mathbb{K} \otimes \mathbb{C}^{n_i})$

CHONG-MAN CHO

Department of Mathematics
College of Natural Science
Hanyang University
Seoul 133, Korea

(Received May 8, 1986)

ABSTRACT. For $1 < p, r \leq \infty$, $X = \left( \bigoplus_{i=1}^{\infty} \mathbb{K} \otimes \mathbb{C}^{n_i} \right)$, $\{n_i\}$ bounded, the space $K(X)$ of all compact operators on $X$ is the only nontrivial M-ideal in the space $B(X)$ of all bounded linear operators on $X$.

KEY WORDS AND PHRASES. Compact operators, hermitian element, M-ideal.

1980 AMS SUBJECT CLASSIFICATION CODE. Primary 46A32, 47B05, secondary 47B05.

1. INTRODUCTION.

Since Alfsen and Effros [1] introduced the notion of an M-ideal, many authors have studied M-ideals in operator algebras. It is known that $K(X)$, the space of all compact operators on $X$, is an M-ideal in $B(X)$, the space of all bounded linear operators on $X$, if $X$ is a Hilbert space or $l_p(1 < p < \infty)$. Smith and Ward [2] proved that M-ideals in a C$^*$-algebra are exactly the closed two sided ideals. Smith and Ward [3], and Flinn [4] proved that, for $1 < p < \infty$, $K(l_p)$ is the only nontrivial M-ideal in $B(l_p)$. The purpose of this paper is to generalize this result to $B(X)$, where $X = \left( \bigoplus_{i=1}^{\infty} \mathbb{K} \otimes \mathbb{C}^{n_i} \right)$, for $1 < p, r \leq \infty$ and $\{n_i\}$ a bounded sequence of positive integers. In this proof, the ideas and results of [4], [2], [5] and [3] are heavily used.

2. NOTATIONS AND PRELIMINARIES.

If $X$ is a Banach space, $B(X)$ (resp. $K(X)$) will denote the space of all bounded linear operators (resp. compact linear operators) on $X$.

A closed subspace $J$ of a Banach space $X$ is an L-summand (resp. M-summand) if there is a closed subspace $\bar{J}$ of $X$ such that $X$ is the algebraic direct sum of $J$ and $\bar{J}$, and $\|x + y\| = \|x\| + \|y\|$ (resp. $\|x\| = \max \{\|x\|, \|y\|\}$) for $x \in J, y \in \bar{J}$. A projection $P: X \to X$ is an L-projection (resp. M-projection) if $\|x\| = \|Px\| + \|(I - P)x\|$ (resp. $\|x\| = \|Px\| + \|(I - P)x\|$) for every $x \in X$. 

\text{(Received May 8, 1986)}
A closed subspace $J$ of a Banach space $X$ is an $M$-ideal in $X$ if $J^\perp = \{ x^* \in X^*: x^*|_J = 0 \}$ is an $l$-summand in $X^*$.

If $(X_i)_{i=1}^\infty$ is a sequence of Banach spaces for $1 \leq p \leq \infty$, $\bigoplus_{i=1}^\infty X_i$ is the space of all sequences $x = (x_i)_{i=1}^\infty$, $x_i \in X_i$, with the norm $\|x\| = \left( \sum_{i=1}^\infty \|x_i\|_p \right)^{1/p} < \infty$ if $1 \leq p < \infty$ and $\|x\| = \sup_i \|x_i\| < \infty$ if $p = \infty$.

An element $h$ in a complex Banach algebra $A$ with the identity $e$ is hermitian if $\|e^{1/2}h\| = 1$ for all real $\lambda$ [6].

If $J_1$ and $J_2$ are complementary nontrivial $M$-summands in $A$ (i.e. $A = J_1 \oplus J_2$), $P$ is the $M$-projection of $A$ onto $J_1$ and $z = P(e) \in J_1$, then $z$ is hermitian with $z = z^2$ [2, 3.1], $zJ_1 \subseteq J_1$ $(i = 1, 2)$ and $zJ_2z = 0$ [2, 3.2 and 3.4]. Since $I - P$ is the $M$-projection of $A$ onto $J_2$, $(e - z) = (e - z)^2$ is hermitian, $(e - z)J_1 \subseteq J_1$ $(i = 1, 2)$ and $(e - z)J_1(e - z) = 0$.

If $M$ is an $M$-ideal in a Banach algebra $A$, then $M$ is a subalgebra of $A$ [2, 3.6]. If $h \in A$ is hermitian and $h^2 = e$, then $hM \subseteq M$ and $Mh \subseteq M$ [4, Lemma 1].

If $A$ is a Banach algebra with the identity $e$, then $A^{**}$ endowed with Arens multiplication is a Banach algebra and the natural embedding of $A$ into $A^{**}$ is an algebra isomorphism into [6]. If $J$ is an $M$-ideal in $A$, then $A^{**} = J^{**} \bigoplus_{\omega}(J^{**})^\perp$ and the associated hermitian element $z \in J^{**}$ commutes with every other hermitian element of $A^{**}$ [5, 2.22].

From now $X$, will always denote $\bigoplus_{i=1}^\infty \bigotimes_{p}^{n_i}$, where $l < p, r < \infty$ and $(n_i)_{i=1}^\infty$ a bounded sequence of positive integers. An operator $T \in B(X)$ has a matrix representation with respect to the natural basis of $X$. From the definition, it is obvious that any diagonal matrix $T \in B(X)$ with real entries is hermitian.

Flinn [4] proved that if $M$ is an $M$-ideal in $B(\ell^p_r)$ and $h \in B(\ell^p_r)$ is a diagonal matrix, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is valid for $X$. He also proved that if $M$ is a nontrivial $M$-ideal in $B(\ell^p_r)$, then $M \nsubseteq K(\ell^p_r)$. Again his proof with a small modification is valid for $X$.

Thus we have observed that if $M$ is a nontrivial $M$-ideal in $B(X)$, then $M \nsubseteq K(X)$.

If $M$ is an $M$-ideal in a Banach algebra $A$ and $h \in M$ is hermitian, then $hAh \subseteq M$.

Indeed, $(e - z)h = (e - z)^2h = (e - z)h(e - z) = 0 = h(e - z)$ and so $zh = hz = h$.

Since $zA^{**}z \subseteq M^{**}$ [2, 3.4], $zAz \subseteq M^{**}$ and hence $hAh = hAzh \subseteq M^{**}$. Since $h \in M$, $hAh \subseteq A \cap M^{**} = M$. Thus if $e \in M$, then $A = M$.

3. MAIN THEOREM.

We may assume that $X = (\ell^1_r \oplus \ell^1_p \oplus \ell^1_r \oplus \ell^1_p \oplus \ldots \oplus \ell^1_r \oplus \ell^1_p \oplus \ldots \oplus \ell^1_r \oplus \ell^1_p \oplus \ldots)$. 

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Set $\alpha = m_1 + \ldots + m_s$ and $\beta = n_1 + \ldots + n_k$. Let $N$ be the set of all natural numbers, $S_0 = \{1, 2, \ldots, \alpha\}$ and, for $1 \leq j \leq k$, $S_j = \bigcup_n (n + \beta N)$, where $n$ runs over $\alpha + n_0 + n_{j-1} < n \leq \alpha + n_0 + \ldots + n_j, \ n \geq 0$. Let $P_j$ be the projection on $X$ defined by $P_j x = 1_{S_j} x$ for every $x \in X$, where $1_{S_j}$ is the indicator function of the set $S_j$. Let $(e_{i1})_{i=1}^\infty$ be the unit vector basis for $X$. $A = \sum_{j=1}^\infty a_{ij} e_j \otimes e_i \in B(X)$ is the operator with matrix $(a_{ij})$ with respect to $(e_{i1})_{i=1}^\infty$.

**LEMMA 1.** If $M$ is an $M$-ideal in $B(X)$ and contains $A$, $a_{ij} e_j \otimes e_i \in B(X)$ such that $(a_{ii})_{i=1}^\infty \subseteq \ell_\infty \setminus c_0$, then $M = B(X)$.

**PROOF.** By multiplying by diagonal matrices from both sides, and as in Lemma 2 [4], we may assume that $A = \sum_{i=1}^\infty e_i f(i) \otimes e_{f(i)}$, where $f(i+1) - f(i) \geq \beta, f(i) \in S_j$ for all $i$ and a fixed $j (1 \leq j \leq k)$. Fix $j (\neq j, 1 \leq l \leq k)$ and $s$

$$(a + n_0 + \ldots + n_{j-1} < s < a + n_0 + \ldots + n_k), \text{ and let } g(i) = s + (i-1)\beta (i = 1, 2, 3, \ldots).$$

**CLAIM:** $B = \sum_{i=1}^\infty e_i g(i) \otimes e_{f(i)} \in M$. Suppose $B \notin M$. Choose $\phi \in M^\perp$ so that $\|\phi\| = 1 = \phi(B)$. Since $\|B\| = 1$ and $AB = B, \forall \in B(X)^*$ defined by $\psi(G) = \phi(GB)$ has norm one and attains its norm at $A \in M$. Hence $\psi \in \tilde{M}$ and $\|\phi + \psi\| = 2$, where $B(X)^* = M^\perp \tilde{M}$. Since $|(\phi + \psi)(G)| = |\phi(G + GB)| \leq \|\phi\| \|G\| \|I + B\|$, $\|\phi + \psi\| \leq \|I + B\|$. To draw a contradiction, we will show that $\|I + B\| < 2$. Let $j$ and $k$ be as above. For $x \in X$ with $\|x\| = 1, \|x\|^p = \|P_j x\|^p + \|(I - P_j) x\|^p$. Let $t = \|P_j x\|^p$, then $1 - t = \|(I - P_j) x\|^p$. Since $Bx$ has support in $S_j$ and $\|Bx\| \leq \|(I - P_j) x\|^p$, we have

$$\| (I + B)x \| \leq 1 + \|Bx\| \leq 1 + (1 - t)^{1/p} \quad (3.1)$$

$$\| (I - P_j)x + Bx \| \leq (2\| (I - P_j)x\|^p)^{1/p} = 2^{1/p}(1 - t)^{1/p}. \text{ Hence}$$

$$\| (I + B)x \| = \| x + Bx \| \leq \| P_j x \| + \| (I - P_j)x + Bx \| \leq t^{1/p} + 2^{1/p}(1-t)^{1/p} \quad (3.2)$$

Obviously, $F(t) = t^{1/p} + 2^{1/p}(1-t)^{1/p}$ is continuous on $[0,1]$ and $F(0) = 2^{1/p} < 2$ so $F(t) < 2$ for all $0 \leq t \leq \delta$. For $\delta < t < 1, 1 + (1 - t)^{1/p} < 2$. By (3.1) and (3.2) above, $\| (I + B) \| < 2$. Contradiction! Hence $B \in M$.

Similarly $C = \sum_{i=1}^\infty e_i f(i) \otimes e_{g(i)} \in M$ (use $\|C\| = 1, CA = C, \psi(G) = \phi(IG), I + C$ is the adjoint of $I + B$). Hence $\|I + C\| < 2$.

Since $M$ is an algebra, $1_{S_0 + \beta N} : I = CB \in M$. Thus for all $i = \alpha + 1, \alpha + 2, \ldots, \alpha + \beta, 1_{S_0 + \beta N} : I \in M$. Since $1_{S_0} : I$ is compact, $1_{S_0} : I \in M$. This proves $M = B(X)$. 

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COROLLARY 2. If $M$ is an $M$-ideal in $B(X)$ and there exists an isometry 
$\tau: B(X) \to B(X)$ so that $\tau(M)$ contains an $A = \sum_{i=1}^{n} e_j \otimes e_i$ with $(a_{ii})_{i>1} \in F_{\varepsilon} \cap O$, 
then $M = B(X)$.

PROOF. Since $\tau(M)$ is an $M$-ideal in $B(X)$ and $A \in \tau(M)$, by the lemma $\tau(M) = B(X)$. Hence $M = B(X)$.

THEOREM 3. If $M$ is an $M$-ideal in $B(X)$ and contains a noncompact $T = \sum_{i=1}^{\infty} e_j \otimes e_i$, 
then $M = B(X)$.

PROOF. Suppose $T \in M$ and $T$ is not compact. Wlog we may assume 
$T = \sum_{k=1}^{\infty} T_k$, $T_k = \sum_{i=m_k+1}^{m_k+n_k} e_j \otimes e_i$, $\|T_k\| = 1$ where $m_k \in \mathbb{N}$, $n_k \in \mathbb{N}$, and 
$m_k + n_k + \beta < m_k + 1$.

Since each $T_k$ has norm one, there exists norm one vectors 
$x_k = (x_{ik}) \in X$, $y_k = (y_{ik}) \in X^*$, $z_k = (z_{ik}) \in X^*$ all with supports $\sigma_k = \{i: m_k \leq i \leq m_k + n_k\}$ so that $y_k(T_kx_k) = \|T_k\| = 1$.

Let $B_k = \sum_{i \geq 1} \sum_{j=m_k+1}^{m_k+n_k} \sum_{j=1}^{\infty} x_{ij} \otimes e_j$, $C_k = \sum_{i \geq 1} \sum_{j=m_k+1}^{m_k+n_k} \sum_{j=1}^{\infty} y_{ij} \otimes e_j \otimes e_{m_k+1}$, 
$D_k = \sum_{i \geq 1} \sum_{j=m_k+1}^{m_k+n_k} \sum_{j=1}^{\infty} z_{ij} \otimes e_j \otimes e_{m_k+1}$, 
$A = \sum_{k \geq 1} e_k \otimes e_{m_k+1}$, $B = \sum_{k \geq 1} B_k$, $C = \sum_{k \geq 1} C_k$ and $D = \sum_{k \geq 1} D_k$. Then all of these operators have norm one and $DB = CTB = A$.

Let $P$ be the matrix obtained from the identity matrix $I$ by interchanging $(m_k + j)$-th column and $(m_k + n_k + j)$-th column for all $k$ and $j(1 \leq j \leq \beta)$. Then $P$ is an isometry in $X$ since $n_k \in \mathbb{N}$.

CLAIM. If $B \in M$, then $M = B(X)$.

Choose $\phi \in c_{\varepsilon} \subseteq F_{\varepsilon}^*$ so that $||\phi|| = 1 = \phi((1,1,1,1,\ldots))$. Define norm one functional 
$\gamma \in B(X)^*$ by $\gamma(G) = \phi((g_{m_k+n_k+1}, m_k+1)_{k \geq 1})$ where $G = \sum_{i=1}^{\infty} g_{ij} \otimes e_i$. Then $\gamma \notin M$. In fact, if $\gamma \in M$, then $\gamma_1 \in B(X)^*$ defined by $\gamma_1(G) = \phi((DG)_{m_k+1}, m_k+1)$ has norm one and attains its norm at $B \in M$. Hence $\gamma_1 \notin M$ and $||\gamma + \gamma_1|| = 2$. But for any norm one $G \in B(X)$, we have 
$$|\gamma + \gamma_1(G)| = |\phi((g_{m_k+n_k+1}, m_k+1) + \sum_{j \geq 1} g_{j} \otimes e_{m_k+n_k+1})_{k \geq 1}|$$
$$\leq \sup_k \|z_k + e_{m_k+n_k+1}\| \|z_k + e_{m_k+n_k+1}\| = 2^1/p'$$
where $\frac{1}{p} + \frac{1}{p'} = 1$. 

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so $\| \gamma + \gamma_1 \| \leq 2^{1/p'}$ contradiction! Thus $\gamma \notin M$. Since $\gamma \notin M$, there is $G \in M$ s.t. $\gamma(G) \neq 0$. So $(g_{m_k+n_k+1, m_k+1}^{m_k+n_k+1})_{k \geq 1} \in \ell^\infty \setminus c_0$. The sequence of the diagonal entries of $P(G)$ belongs to $\ell^\infty \setminus c_0$. Thus by corollary 2, $M = B(X)$. This proves the claim.

Next $\Psi \in B(X)^*$ defined by $\Psi(G) = \Phi((a_{m_k+n_k+1, m_k+1}^{m_k+n_k+1})_{k \geq 1})$ is not in $M$. Indeed, if $\Psi \in M$, then since $I \in B(X)^*$ defined by $I(G) = ((CG)_{m_k+1, m_k+1})_{k \geq 1}$ has norm one and attains its norm at $T \in M$, $\Psi \notin M$ and so $\| \Psi + \Psi_1 \| = 2$. But for any norm one $G \in B(X)$, we have

$$\| (G + \gamma_1)_{G1} \| \leq \sup_{k} \| (CG)_{m_k+n_k+1, m_k+1}^{m_k+n_k+1} \| + \sum_{j \in \mathcal{O}} (CG)_{m_k+1, m_j} + \xi_j$$

$$\leq \sup_k \| x_k + e_{m_k+n_k+1} \|$$

since $CG \in B(X)$, $\| CG \| = 1$

$$= 2^{1/p}, \text{ contradiction!}$$

Thus $\Psi \notin M$. So there is $G = \sum_{j \in \mathcal{O}} e_j \Phi e_j \in M$ such that $(a_{m_k+n_k+1, m_k+1}^{m_k+n_k+1})_{k \geq 1} \in \ell^\infty \setminus c_0$.

There is $\epsilon > 0$ such that $\| G_k \| > \epsilon$ for infinitely many $k$, where

$$G_k = \sum_{j \in \mathcal{O}} e_j, m_k+n_k+1 e_j.$$ We can choose diagonal matrices $D_1$ and $D_2$ in $B(X)$ so that $D_1 G D_2$ has the same form as $B$ in the claim above. Since $D_1 G D_2 \in M$, $M = B(X)$.

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