ON SOME PROPERTIES OF POLYNOMIAL RINGS

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(Received February 4, 1986 and in its revised form April 29, 1986)

ABSTRACT. For a commutative ring with unity R, it is proved that R is a PF-ring if and only if the annihilator, \( \text{ann}(a) \), for each \( a \in R \) is a pure ideal in \( R \). Also it is proved that the polynomial ring, \( R[X] \), is a PF-ring if and only if \( R \) is a PF-ring. Finally, we prove that \( R \) is a PP-ring if and only if \( R[X] \) is a PP-ring.

KEY WORDS AND PHRASES. Polynomial Rings, Pure ideal, PF-ring, PP-ring, \( R \)-flatness, and idempotent elements.

1980 AMS SUBJECT CLASSIFICATION CODE: 13B.

1. INTRODUCTION.

All our rings in this paper are commutative with unity. An ideal \( I \) of a ring \( R \) is called pure if for any \( x \in I \), there exists \( y \in I \) such that \( xy = x \). A ring is called a PF-ring if every principal ideal \( aR \) is a flat \( R \)-module. A ring \( R \) is called a PP-ring if every principal ideal \( aR \) is a projective \( R \)-module. One can easily show that \( aR \) is projective if and only if the annihilator, \( \text{ann}(a) \), is generated by an idempotent element, (see [1], [2]).

First, we state a proposition characterizing flat \( R \)-modules elementwise. This is a well known result in commutative ring theory, (see [3]).

PROPOSITION 1. An \( R \)-module \( M \) is a flat \( R \)-module if and only if for any pair of finite subsets \( \{x_1, x_2, \ldots, x_n\} \) and \( \{a_1, a_2, \ldots, a_n\} \) of \( M \) and \( R \) respectively, such that \( \sum_{i=1}^{n} x_i a_i = 0 \) there exists elements \( z_1, \ldots, z_k \in M \) and \( b_{ij} \in R \); \( i = 1, 2, \ldots, k \), such that \( \sum_{j=1}^{k} b_{ij} a_i = 0 \), \( j=1, 2, \ldots, k \), and \( x_i = \sum_{j=1}^{k} z_j b_{ij} \), \( i = 1, 2, \ldots, n \).

In the following theorem we establish that \( R \) is a PF-ring if and only if \( \text{ann}(a) \) for each \( a \in R \) is a pure ideal.

THEOREM 1. For any ring \( R \), \( R \) is a PF-ring if and only if \( \text{ann}(m) \) for each \( m \in R \) is a pure ideal.

PROOF. Let \( x_1, x_2, \ldots, x_n \in mR \) and \( a_1, a_2, \ldots, a_n \in R \) with \( \sum_{i=1}^{n} x_i a_i = 0 \). Then there exists \( m_1, m_2, \ldots, m_n \in R \) such that \( x_i = m_i m_i \), \( i = 1, 2, \ldots, n \). So \( \sum_{i=1}^{n} m_i a_i = 0 \). Hence \( m \in \text{ann}(\sum_{i=1}^{n} m_i a_i) \).
Since \( \text{ann}(\sum_{i=1}^{n} m_{i}a_{i}) \) is a pure ideal, there exists \( b \in \text{ann}(\sum_{i=1}^{n} m_{i}a_{i}) \) such that \( bm = m \).

Now take \( m \in mR \) and \( bm_{1}, bm_{2}, \ldots, bm_{n} \in R \). These elements satisfy \( \sum_{i=1}^{n} bm_{i}a_{i} = 0 \) and \( bm_{1}m = m_{1}m = x_{1}, i = 1, 2, \ldots, n \). Therefore \( mR \) is a flat \( R \)-module.

Conversely, let \( b \in \text{ann}(m) \). Then \( mb = 0 \). Since \( bR \) is a flat \( R \)-module, there exists \( c \in bR \) such that \( dm = 0 \) and \( b = cd \). Now \( c = c_{1}b \), so \( b = cd = c_{1}d \). Moreover \( c_{1}d \in \text{ann}(m) \). Therefore \( \text{ann}(m) \) is a pure ideal.

**Lemma 1.** Let \( I_{1}, I_{2}, \ldots, I_{n} \) be a finite set of pure ideals of a ring \( R \), then \( \bigcap_{j=1}^{n} I_{j} \) is a pure ideal.

**Proof.** Let \( x \in \bigcap_{j=1}^{n} I_{j} \). Then \( x \in I_{j} \) for each \( j \). Thus there exists \( y_{1} \in I_{1}, y_{2} \in I_{2}, \ldots, y_{n} \in I_{n} \) with \( xy_{j} = x, j = 1, 2, \ldots, n \). Then \( y = y_{1}y_{2} \ldots y_{n} \in J \) and \( xy = x \).

Let \( R \) be a reduced (without nonzero nilpotent elements) ring. Let \( h(X) = h_{0} + h_{1}X + \ldots + h_{n}X^{n} \in R[X] \). Then \( \text{ann}(h(X)) \subseteq N[X] \), where \( N \) is the annihilator of the ideal generated by \( h_{0}, h_{1}, \ldots, h_{n} \), that is \( N = \text{ann}(h_{0}, h_{1}, \ldots, h_{n}) \) and \( \bigcap_{i=0}^{n} \text{ann}(h_{i}) \).

Moreover if \( f(X) = a_{0} + a_{1}X + \ldots + a_{m}X^{m} \in \text{ann}(h(X)) \) then \( a_{i}h_{j} = 0 \) for all \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) (see [4]).

**Lemma 2.** Let \( R \) be a PF-ring, then \( R \) is reduced.

**Proof.** Let \( a \) be a nilpotent element in \( R \), \( a \neq 0 \). Let \( n \) be the least positive integer greater than 1 such that \( a^{n} = 0 \). Hence \( a \in \text{ann}(a^{n-1}) \). Since \( \text{ann}(a^{n-1}) \) is pure, there exists \( b \in \text{ann}(a^{n-1}) \) with \( ab = a \). Now \( o = ba^{-1} = a^{-1} \) since \( ba = a \). Contradiction. Thus \( R \) is reduced.

**Theorem 2.** The ring of polynomials, \( R[X] \), is a PF-ring if and only if \( R \) is a PF-ring.

**Proof.** Let \( f(X) = a_{0} + a_{1}X + \ldots + a_{m}X^{m} \in \text{ann}(h(X)) \) where \( h(X) = h_{0} + h_{1}X + \ldots + h_{n}X^{n} \in R[X] \).

Since \( R[X] \) has no nonzero nilpotent elements, \( a_{i} \in J = \bigcap_{j=0}^{n} \text{ann}(h_{j}) \), \( i = 0, 1, 2, \ldots, m \).

By Lemma 1, \( J \) is pure. Hence there exist \( b_{1}, b_{2}, \ldots, b_{m} \in J \) such that \( a_{i}b_{i} = a_{i} \) if \( i = 1, 2, \ldots, m \). Now our aim is to find \( c \in J \) such that \( c f(X) = f(X) \). We construct this element inductively.

First, \( a_{0}b_{0} = a_{0} \). Consider

\[
( a_{0} + a_{1}X ) ( b_{0} + b_{1} - b_{1}b_{0} ) = a_{0}b_{0} + a_{0}b_{1} - a_{0}b_{1}b_{0} + a_{1}b_{0}X + a_{1}b_{1}X - a_{1}b_{0}b_{1}X
\]

\[
= a_{0} + a_{0}b_{1} - a_{0}b_{1} + a_{1}b_{0} + a_{1}b_{0}X + a_{1}X - a_{1}b_{0}X
\]

\[
= a_{0} + a_{1}X.
\]

Let \( c_{1} = b_{0} + b_{1} - b_{1}b_{0} \), then

\[
( a_{0} + a_{1}X + a_{2}X^{2} ) ( c_{1} + b_{2} - c_{1}b_{2} ) = (a_{0} + a_{1}X)c_{1} + b_{2}(1 - c_{1})(a_{0} + a_{1}X) + a_{2}c_{1}X^{2} + a_{2}b_{2}X^{2} - a_{2}b_{2}c_{1}X^{2}
\]
\[ a_0 + a_1 X + a_2 c_1 X^2 + a_2 b_2 X^2 - a_2 c_1 X^2 = a_0 + a_1 X + a_2 X^2 \]

Similarly, \( c_2 = c_1 + b_2 - c_1 b_2, \ldots \)

\[ c_m = c_{m-1} + b_m - c_{m-1} b_m \] and

\[ (a_0 + a_1 X + \ldots + a_1 X^i) c_i = a_0 + a_1 X + \ldots + a_1 X^i \]

\[ i = 0, 1, 2, \ldots, m. \] Moreover \( c_0, c_1, \ldots, c_m \in J. \)

Thus there exist \( c = c_m \in J \) with \( cf(X) = f(X). \)

Conversely, assume \( R[X] \) is a PF-ring. Let \( a \in R \) and \( b \in \text{ann}(a). \)

Then \( b \in \text{ann}(a). \) Since \( R \) is a PF-ring there exists

\[ g(X) = c_0 + c_1 X + \ldots + c_k X^k \in \text{ann}(a) \]

with \( b \) \( g(X) = b. \) Hence \( bc_0 = b \) and \( c_0 a = 0. \)

Consequently, \( R \) is a PF-ring.

**THEOREM 3.** \( R \) is a PP-ring if and only if \( R[X] \) is a PP-ring.

**PROOF.** It is enough to show that \( \text{ann}(f(X)) \) is generated by an idempotent \( R[X] \)

element in \( R[X], \) where \( f(X) = a_0 + a_1 X + \ldots + a_n X^n. \) Since \( R \) is reduced,

\[ \text{ann}(f(X)) = N[X] \] where \( N \) is the annihilator of the ideal generated by \( R[X] \)

\[ a_0, a_1, \ldots, a_n. \]

\[ N = \text{ann}(a_0, a_1, \ldots, a_n) \]

\[ = \bigoplus_{i=0}^{n} \text{ann}(a_i) \]

\[ = \bigoplus_{i=0}^{n} e_i R, \quad e_i^2 = e_i \text{ because } R \text{ is a PP-ring.} \]

\[ = (e_1 e_2 \ldots e_n) R \]

\[ = e R, \] where \( e = e_1 e_2 \ldots e_n. \)

Hence \( \text{ann}(f(X)) = e R[X], \) \( e^2 = e \)

R[X]

Conversely, let \( R[X] \) be a PP-ring, let \( a \in R, \) then consider \( \text{ann}(a). \) Since \( R[X] \)

is a PP-ring, \( \text{ann}(a) = g(X) R[X], \) where \( g(X) = g(X). \) If \( g(X) = b_0 + b_1 X + \ldots + b_m X^m, \)

then \( b_0^2 = b_0. \) We claim \( \text{ann}(a) = b_0 R. \) Let \( b \in \text{ann}(a), \) then \( ba = 0. \) So \( b \in g(X) R[X]. \)

Thus \( b = (b_0 + b_1 X + \ldots + b_m X^m)(c_0 + c_1 X + \ldots + c_t X^t). \) Therefore \( b = b_0 c_0, \) that is \( b \in b_0 R. \)

For the other way around, let \( b \in b_0 R. \) Then \( b = b_0 c_0 \) for some \( c_0 \in R. \) Since \( b_0 a = 0. \) That is \( b \in \text{ann}(a). \) Thus \( \text{ann}(a) = b_0 R. \)
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