ON COMPLEX $L_1$-PREDUAL SPACES

MRINAL KANTI DAS
Department of Mathematics
University of Nairobi
P.O. Box 30197, Nairobi, Kenya.

(Received November 12, 1985 and in revised form April 30, 1986)

ABSTRACT - This paper contains some characterisations of complex $L_1$-balls, including interpolation theorems which are analogs of Edward's separation theorem for simplices.

KEY WORDS AND PHRASES: Boundary measures, Choquet ordering, Haar measure, Lindenstrauss space, Simplex.

1980 AMS SUBJECT CLASSIFICATION CODE. 46B99.

1. INTRODUCTION.

The purpose of this paper is to furnish some characterisations of complex $L_1$-predual spaces which are being known as Lindenstrauss spaces after [1]. The dual unit balls of such spaces now-a-days called $L_1$-balls have been characterised by many authors including Lazar & Lindenstrauss [2], Lazar [3], [4], Lau [5] and others when the spaces are real. But their complex versions far from being trivial follow-ups seem to be much complicated and in reality sometimes need ingenuity to be formulated even. This paper contains some complex versions of Lau's results [5] embodied in Theorem 3.

2. NOTATIONS AND PRELIMINARIES

For a compact convex subset $K$ of a locally convex Hausdorff space $E$, $\partial_e K$ stands for the set of its extreme points; $M(K)$ for the Banach space (with total variation as norm) of complex regular Borel measures on $K$; $M^1(K)$ for the set of members of $M(K)$ with norm $\leq 1$; $C(K)$, $A(K)$, $P(K)$ for the space of all real-valued continuous functions, continuous affine functions, continuous convex functions on $K$ respectively.

For bounded real-valued functions $f$ on $K$, the upper envelope is denoted by $\hat{f}$ and the lower envelope by $\check{f}$. A measure $\mu$ is said to be a boundary measure if $|\mu|$ is maximal in the ordering of Choquet; in fact $\mu$ is a boundary measure iff $\mu(\check{f} - f) = 0$ for all $f \in C(K)$ [6, p.129]. We shall also write $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$.

If $V$ is a complex Banach space, the dual unit ball $K = (V^*)^1$ is convex and compact in the $w^*$-topology. We define the map $\hom f$ as $(\hom f)(x) = \int_{\partial_e K} f(ux) \, d\mu$ for semi-continuous function $f$ on $K$, where $d\mu$ is the unit Haar measure on $\Gamma$. Clearly $\hom f$ is $\Gamma$-homogeneous, i.e. $(\hom f)(\beta x) = \beta (\hom f)(x)$ for $\beta \in \Gamma$. One can easily show that $\hom$ restricted to $C(K)$ are norm-decreasing projections of $C(K)$ onto the space of $\Gamma$-homogeneous
continuous functions on $K$. The adjoint projection horn defined as $\text{horn} = \sigma \text{hom}$ is also a norm decreasing $w^*$-continuous projection of $M(K)$ onto a linear subspace $M_{\text{hom}}(K)$ of $M(K)$. We can write $(\text{horn} f)(x) = S_1 f(x) + i T f(x)$
where $S_1 f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \theta f(xe^{i\theta}) d\theta$, $T f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \theta f(xe^{i\theta}) d\theta$.

If we write $(S f)(x) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos \theta f(xe^{i\theta}) d\theta$
(which multiplied by $\pi$ gives what Roy [7] has defined as $f_+$), then
$2(S_1 f)(x) = \text{odd} S f(x)$.

Throughout the paper we shall write $A_o(K)$ for the set of all continuous $\Gamma$-homogeneous affine functions on $K = (V^*)_1$.

3. Main Results

For real Banach space $V$, the following results are recently proved.

**Theorem 1.** If $K$ is the dual unit ball of a real Banach space $V$, then the following are equivalent:

(i) $V$ is an $L_1$-predual space;

(ii) If $\mu_1, \mu_2$ are boundary measures on $K$ having the same barycentre, then $\text{odd } \mu_1 = \text{odd } \mu_2$.

(iii) For $f \in P(K)$, $\text{odd } f$ is affine.

(iv) For any $f \in P(K)$, $\hat{f}(0) = \frac{1}{2} \sup \{f(x) + f(-x) : x \in K\} = \sup \{\text{even } f(x) : x \in K\}$.

(v) For any l.s.c. concave function $f$ on $K$ such that even $f \geq 0$, there exists a continuous affine symmetric function $a$ on $K$ such that $f \geq a$.

(vi) If $f, -h$ are l.s.c. concave functions on $K$ such that $h \leq f$ and $\sup \{\text{even } h(x) : x \in K\} \leq \inf \{\text{even } f(x) : x \in K\}$, then these exist a continuous affine symmetric function $a$ on $K$ such that $h \leq a \leq f$.

The equivalence of (i) - (iv) is due to Lazar [4] while that of (i), (v), (vi) is proved by Lau [5].

Many interesting developments are noted when efforts are made to obtain complex analogs of these results (many others not stated here) of real Lindenstrauss spaces. A brilliant step towards this have been made by Effros [8] who has shown that odd $\hat{\mu}$ is to be replaced by hom $\mu$ in complex space. Olsen [9] has shown that the hypothesis even $f \geq 0$ in (v) is to be replaced by $\Sigma f(\zeta_k, x) \geq 0$ for $\zeta_k \in \Gamma$ with $\Sigma \zeta_k = 0$, $x \in K$. Subsequently Roy [7] has tried to give complex analog of (iv), replacing odd $\hat{f}$ by odd $(S f)^\wedge$. His formulation is rather partial. But [9] contains some interesting examples.

Below we give a characterisation of complex $L_1$-predual space $V$ which is a kind of complex analog of Lau's result and is due to Olsen [9].

**Theorem 2.** If $K$ is the dual unit ball of a complex Banach space $V$, then the following are equivalent:

(i) $V$ is an $L_1$-predual space;

(ii) For every l.s.c. concave function $f$ on $K$ such that $\Sigma f(\zeta_k, x) \geq 0$ for all $x \in K$ and $\zeta_k \in \Gamma$, $k = 1, 2, \ldots, n$ with $\Sigma \zeta_k = 0$, there is an $a \in A_o(K)$ such that $\text{re } a(x) \leq f(x)$ for all $x \in K$.

We give in Theorem 3, some complex analogs of Lau's result. However to start with, we furnish a Lemma below:
Lemma 1. If \( \mu \) be a non-zero positive measure on a compact convex subset \( K \) of a locally convex Hausdorff space \( E \), then for all u.s.c. convex function \( f \) on \( K \)

\[
f(x) \leq \mu(K)^{-1} \int f(y) \, d\mu \quad \text{where } r(\mu) = x.
\]

Proof: By a well-known result [10; I.2,2], the stated inequality holds for \( f \in P(K) \). Now applying Mokobodzki([10; I.5,1]) that for every u.s.c. convex function \( f \), there is a descending net \( \{ f_\alpha : f_\alpha \in P(K) \} \) which converges to \( f \), we get the desired result.

Our main result is

Theorem 3. If \( K \) is the dual unit ball of a complex Banach space \( V \), then the following are equivalent:

(i) \( V \) is an \( L_1 \)-predual space;

(ii) If \( f \) is a l.s.c. concave function on \( K \) with even \( Sf(x) \geq 0 \) for all \( x \in K \), then there exists an \( a \in A_0(K) \) such that \( \Re a \leq f \) on \( K \);

(iii) If \( a, -h \) are l.s.c. concave functions on \( K \) such that \( h \leq f \) and even \( S h(x) \leq 0 \), then there exists an \( a \in A_0(K) \) such that \( \Re a \leq f \) on \( K \).

(iv) If \( a, -h \) are l.s.c. concave functions on \( K \) such that \( h \leq f \) and

\[
\sup_{k=1}^n \{ \sum_{k=1}^n \alpha_k f_\kappa(x) : x \in K, \alpha_k \leq 1, \sum_{k=1}^n \alpha_k = 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0 \}
\]

\[
\leq 0 \quad \text{inf}_{k=1}^n \{ \sum_{k=1}^n \alpha_k f_\kappa(x) : x \in K, \alpha_k \leq 1, \sum_{k=1}^n \alpha_k = 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0 \};
\]

then there is an \( a \in A_0(K) \) such that \( \Re a \leq f \) on \( K \);

(v) If \( g \) is an u.s.c. convex function on \( K \), then \( g(0) \leq \sup \{ \sum_{k=1}^n \alpha_k g_\kappa(x) : x \in K, \alpha_k \leq 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0 \} \).

Proof. (i) \( \rightarrow \) (ii).

We shall, in fact, show that (ii) is implied by Theorem 2 (ii). So let \( f \) be l.s.c. concave on \( K \) such that even \( Sf(x) \geq 0 \). We define

\[
F(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left[ \cos \theta f(x e^{i\theta}) \right] \, d\theta
\]

Then \( F(x) = 2 \sum f(x) \). Clearly \( F \) is l.s.c. concave. Let \( \zeta_k \in \Gamma \) for \( k = 1, 2, \ldots, n \) be such that \( \sum \zeta_k = 0 \). Now note that \( Sf(x) = S_1 f(x) + \text{even } Sf(x) \) and that \( \sum \text{Hom} f(\zeta_k x) = 0 \). Thus

\[
2F(\zeta_k x) = 2 \sum \text{even } Sf(\zeta_k x) = 2 \sum \frac{1}{2} f(2 \zeta_k x) + 2 \sum \text{even } Sf(\zeta_k x) =
\]

\[
2 \sum \text{even } Sf(\zeta_k x) \quad \text{which is } \geq 0 \text{ by hypothesis. Consequently by Theorem 2 (ii), there is a function } b \in A_0(K) \text{ such that } \Re b \leq F \text{ on } K.
\]

We consider the measure

\[
\mu = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta \in \{e^{-i\theta} \}
\]

where \( \delta(y) \) is the Dirac measure at \( y \). By [6; p. 115], \( r(\mu) = x \). Also \( \mu(k) = \frac{4}{\pi} \).

On applying Lemma 1, we have \( 2F(x) \leq \mu(K) f(x) \) i.e. \( F(x) \leq \frac{2}{\pi} f(x) \). Putting \( a = \frac{\pi b}{2} \), we have \( \Re a \leq f \) on \( K \).
(ii) \ implies (iii). Let \( f, -h \) be l.s.c., concave functions on \( K \) such that \( h \leq f \) and even \( S h(x) \leq 0 \) for all \( x, y \in K \).

We first show that \( h(x) + h(-x) \leq 0 \leq f(y) + f(-y) \) for all \( x, y \in K \). Let us establish the last inequality, since the first one can be done similarly. To do so we take the measure

\[
\mu = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos \theta \, e(xe^{i\theta}) \, d\theta
\]

and find as before by [6, p. 115] that \( r(\mu) = x, \mu(K) = \frac{4}{\pi} \).

Now apply Lemma 1 to get \( f(y) + f(-y) \geq 0 \) from even \( Sf(y) \geq 0 \).

Now we define \( F(x) = f(x) - h(-x) \).

Then \( F \) is l.s.c., concave. Moreover by the hypothesis and the inequalities just proved, we have \( F(x) + F(-x) \geq 0 \) for all \( x \in K \) so that even \( SF(x) \geq 0 \).

By (ii) then we have an \( a \in A_0(K) \) such that \( re a \leq F \). This \( a \) is, in fact, the function with the desired property.

(iii) \ implies (iv). Let \( f, -h \) be two l.s.c., concave functions on \( K \), which satisfy the conditions given in the hypothesis of (iv). Then clearly even \( Sh(x) \geq 0 \) for all \( y \in K \).

(iv) \ implies (v). Suppose that \( g \) is an u.s.c., convex function and let

\[
g_o = \sup \{ \alpha_k g(\zeta_k x) \mid x \in K, \alpha_k \geq 0, \Sigma \alpha_k = 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0 \}
\]

We assume that \( g_o = 0 \); there will be no loss of generality in the assumption since \( (g + \alpha)^\wedge (O) = \hat{g} + O + \alpha \)

for positive real number \( \alpha \).

We define \( F = -\hat{g} \) where \( (\hat{g})(x) = g(-x) \). Clearly \( F \) is l.s.c., concave.

Since \( g + \hat{g} \leq 2g_o \) by the definition of \( g_o \), it follows that \( g \leq F \). Moreover for \( n \in N, \alpha_k \geq 0, \Sigma \alpha_k = 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0, x \in K \), we have \( \Sigma_k F(\zeta_k x) = -\Sigma_k g(-\zeta_k x) \)

so that \( \inf \{ \Sigma_k F(\zeta_k x) \mid x \in K, \alpha_k \geq 0, \Sigma \alpha_k = 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0 \} = g_o = 0 \).

Thus by (iv), there is an \( h \in A_0(K) \) such that \( g \leq re h \leq F \). We put \( re h = a \in A(K) \).

Now since \( g \leq a \), we have \( \hat{g} = \alpha(O) \leq a(O) \). Again \( -g \geq -2g_o \in A(K) \), so that \( a(O) - 2g_o \leq (\hat{g} = \hat{g} + o) \).

Thus \( \hat{g} = \alpha(O) \leq \hat{g} + 2g_o \) so that \( \hat{g} = \hat{g} + o \) and the result follows.

(v) \ implies (i) is the same as [7; p. 101].

Note: Our result in (v) is sharper than Roy's result [7; Thm 3.3 (iii)] that for \( g \in P(K), \hat{g} = \sup \Sigma_k g(\zeta_k x) \leq K \), \( n \in N, \alpha_k \geq 0, \Sigma \alpha_k = 1, \zeta_k \in \Gamma, \Sigma \alpha_k \zeta_k = 0 \).

In fact this follows from (v) immediately, since the reverse inequality is evident from the concave character of \( \hat{g} \).
In analog with Lazar's selection theorem for Choquet simplex, Lazar & Lindenstrauss formulated a selection theorem for real L-balls which was followed by a complex version by Olsen. Our results which are chiefly complex analogue of Lau's result seem to resemble Edward's interpolation theorem for simplices.

REFERENCE

5. LAU, Ka-Sing : The Dual Ball of Lindenstrauss Space, Math. Scand. 33, (1973), 323-337.