ON THE AFFINE WEYL GROUP OF TYPE $\tilde{A}_{n-1}$

MUHAMMAD A. ALBAR
Department of Mathematical Sciences
University of Petroleum and Minerals
Dhahran, Saudi Arabia

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ABSTRACT. We study in this paper the affine Weyl group of type $\tilde{A}_{n-1}$, [1]. Coxeter [1] showed that this group is infinite. We see in Bourbaki [2] that $\tilde{A}_{n-1}$ is a split extension of $S_n$, the symmetric group of degree $n$, by a group of translations and of a lattice of weights. $\tilde{A}_{n-1}$ is one of the crystallographic Coxeter groups considered by Maxwell [3], [4].

We prove the following:

THEOREM 1. $\tilde{A}_{n-1}$, $n \geq 3$ is a split extension of $S_n$ by the direct product of $(n-1)$ copies of $Z$.

THEOREM 2. The group $\tilde{A}_2$ is soluble of derived length 3, $\tilde{A}_3$ is soluble of derived length 4. For $n > 4$, the second derived group $\tilde{A}''_{n-1}$ coincides with the first $\tilde{A}'_{n-1}$ and so $\tilde{A}_{n-1}$ is not soluble for $n > 4$.

THEOREM 3. The center of $\tilde{A}_{n-1}$ is trivial for $n \geq 3$.

KEY WORDS AND PHRASES. Presentation, Reidemeister-Schreier method, Coxeter group.

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1. INTRODUCTION.

Consider the presentation

$$\tilde{A}_{n-1} = \langle y_1, y_2, \ldots, y_n | y_i^2 = e \text{ if } 1 \leq i \leq n, \quad y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1} \text{ if } 1 \leq i \leq n-1, \quad y_i y_j = y_j y_i \text{ if } 1 \leq i < j \leq n \text{ and } (i,j) \neq (1,n), \quad y_1 y_n y_1 = y_n y_1 y_n \rangle$$

where $n \geq 3$.

This is an irreducible Coxeter group whose graph is a polygon with $n$ vertices. Using some geometrical methods Coxeter showed that $\tilde{A}_{n-1}$ is infinite [4]. This group is also a Weyl group [1]. It is the affine Weyl group of type $\tilde{A}_{n-1}$. We see in Bourbaki [2] that $\tilde{A}_{n-1}$ is a split extension of $S_n$, the symmetric group of degree $n$, by a
group of translations and of a lattice of weights. This group was also considered by Maxwell [3], [4].

The purpose of this paper is to prove that $\tilde{A}_{n-1}$ is a split extension of $S_n$ by a direct product of $(n-1)$ copies of $Z$. The method depends on presentations of group extension [5]. We also find that $\tilde{A}_3$ is soluble of derived length 3, $\tilde{A}_4$ is soluble of derived length 4 and that the second derived group $\tilde{A}_{n-1}''$ coincides with the first $\tilde{A}_{n-1}'$ if $n > 4$ and hence $\tilde{A}_{n-1}$ is not soluble in this case. We finally show that the center of $\tilde{A}_{n-1}$ is trivial.

2. THE STRUCTURE OF $\tilde{A}_{n-1}$.

We show in this section that $\tilde{A}_{n-1}$ is a split extension of $S_n$ by the direct product of $(n-1)$ copies of $Z$. We achieve this by using the method in [5] as follows. We find an epimorphism $\theta: \tilde{A}_{n-1} \rightarrow S_n$ such that the extension

$$1 \longrightarrow \ker\theta \longrightarrow \tilde{A}_{n-1} \longrightarrow S_n \longrightarrow 1 \quad (2.1)$$

splits. It will be required to find a presentation for $\ker\theta$. We guess that it will be isomorphic to $A = Z^{n(n-1)}$ (given by generators and relations). We then construct a new short exact sequence (2.3), where $A$ is embedded as normal subgroup of a group $E$ in such a way that $A$ is the kernel of an epimorphism $\theta': E \rightarrow G$.

$$1 \longrightarrow \ker\theta \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \quad (2.2)$$

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1 \quad (2.3)$$

Then we use Tietze transformations to identify $E$ with $\tilde{G}$, i.e., to find an isomorphism $\phi: E \rightarrow \tilde{G}$, which makes the right-hand square commute. It then follows that $A = \ker\theta$. A presentation for the symmetric group of degree $n \geq 2$ is

$$S_n = \langle x_1, \ldots, x_{n-1} \mid x_i^2 = e \text{ if } 1 \leq i \leq n-1, x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ if } 1 \leq i \leq n-2, x_i x_j = x_j x_i \text{ if } 1 \leq i < j-1 \leq n-1 \rangle.$$ 

We define the mapping $\theta: \tilde{A}_{n-1} \rightarrow S_n$ by

$$\theta: y_i \rightarrow x_i \text{ if } 1 \leq i \leq n-1$$

$$y_n \longrightarrow x_1 x_2 \ldots x_{n-2} x_{n-1} x_{n-2} \ldots x_2 x_1.$$ 

Then $\theta$ is an epimorphism. If $\alpha$ is the mapping from $S_n$ to $\tilde{A}_{n-1}$ defined by

$$\alpha: x_i \rightarrow y_i \text{ if } 1 \leq i \leq n-1,$$

then $\alpha$ is a homomorphism and $\alpha\theta = 1_{S_n}$.

Thus the extension

$$1 \longrightarrow \ker\theta \longrightarrow \tilde{A}_{n-1} \longrightarrow S_n \longrightarrow 1 \quad (\alpha)$$

splits.

We construct the short exact sequence

$$1 \longrightarrow A \longrightarrow E \longrightarrow S_n \longrightarrow 1.$$
A presentation of $E$ will be

$$E = \langle \text{generators of } A, \text{ generators of } S_n \rangle,$$

relations of $A$, relations of $S_n$, action of $S_n$ on $A$ \[ [6] \].

Let $A = \langle a_1, \ldots, a_{n-1} \rangle; a_1 a_k = a_k a_1$ if $1 \leq i \leq k < n-1$ \hspace{1cm} (2.4)

We define the action of $S_n$ on $A$ as follows:

$$x_i = a_1^{-1}$$ \hspace{1cm} (2.5)

$$x_i = a_1^{-1} a_i \text{ if } 2 \leq i \leq n-1$$ \hspace{1cm} (2.6)

$$x_i = \begin{cases} a_{k+1} & \text{if } i = k+1, 1 \leq k < n-1 \\ a_k & \text{if } i = k, 2 \leq i \leq n-1 \\ a_k & \text{otherwise} \end{cases}$$ \hspace{1cm} (2.7-2.9)

**NOTATION.** We let $A_1 = x_2 x_3 \ldots x_i$. We also denote the relations $xyx = yxy$ and $ab = ba$ by $(x,y)$ and $[a,b]$ respectively.

To reduce the relations of $E$ to a manageable form we consider the following lemma and proposition.

**LEMMA 1.** In the group $S_n$ the following identities hold:

(i) $\Delta_k x_i = x_{i+1} \Delta_k$ if $2 \leq i < k$

(ii) $\Delta_k x_i = \Delta_k x_i$ if $i = k$

(iii) $\Delta_k x_i = \Delta_k x_i$ if $i = k+1$

(iv) $\Delta_k x_i = x_i \Delta_k$ if $i > k+1$

(v) $\Delta_k \Delta_i = x_3 \ldots x_i \Delta_i + \Delta_k$ if $2 \leq i \leq k$

(vi) $\Delta_i^2 = x_3 \ldots x_i \Delta_i - 1$.

**PROOF.** (i) $\Delta_k x_i = x_2 x_3 \ldots x_i x_i x_{i+1} x_i x_{i+1} \ldots x_k x_i$

$$= x_2 \ldots x_i x_i x_{i+1} x_i \ldots x_k$$

$$= x_2 \ldots x_i (x_i x_{i+1} x_i x_{i+1} \ldots x_k$$

$$= x_i \Delta_k.$$

(ii) to (iv) obvious.

(v) and (vi) application of (i).

**PROPOSITION 1.** In the group $E$, relations (2.4) to (2.9) become the following:

(i) Relation (2.5) is equivalent to $(a_1 x_i)^2 = e$.

(ii) Relation (2.7) is equivalent to $a_i = a_i^{-1} 2 \leq i \leq n-1$.

(iii) Relation (2.6) is equivalent to $(a_1 x_i, x_2)$.

(iv) Relation (2.8) follows from (ii).

(v) Relation (2.9) is equivalent to $[a_i, x_i]$ for $3 \leq i \leq n-1$.

(vi) Relation (2.4) is equivalent to $(x_2 a_i)^2 = (a_1 x_2)^2$. 
PROOF. (i) Obvious

(ii) Easy by induction on \( i \).

(iii) Using part (ii) relation (2.6) becomes
\[
x_i \Delta_i^{-1} a_i \Delta_i x_i = a_i \Delta_i^{-1} a_i \Delta_i.
\]
Using relation (2.9) it reduces to \((a_1 x_1, x_2)\).

(iv) Obvious by using part (ii).

(v) Using part (ii) relation (2.9) becomes
\[
\Delta_k x_i \Delta_k^{-1} a_i = a_i \Delta_k \Delta_i \Delta_k^{-1}, \quad i \neq k, \ i \neq k+1.
\]
If \( i > k+1 \), then by Lemma 1 (iv) we get
\[
[x_i, a_j] \text{ for } 3 \leq i \leq n-1.
\]
If \( i < k \) then by Lemma 1 (i), we get
\[
[x_{i+1}, a_j] \text{ for } 2 \leq i \leq n-1.
\]
Therefore relation (2.9) is equivalent to \([a_1, x_i]\) for \(3 \leq i \leq n-1\).

(vi) Using part (ii) relation (2.4) becomes
\[
\Delta_k \Delta_k^{-1} a_i \Delta_k \Delta_k^{-1} a_i = a_i \Delta_k \Delta_i \Delta_k \Delta_k^{-1}, \quad 1 \leq i < k \leq n-1.
\]
Using Lemma 1 (v) and relation (2.9), we get
\[
(x_2 a_1)^2 = (a_1 x_2)^2.
\]

THEOREM 1. The group \( E \) is isomorphic to \( \tilde{A}_{n-1} \) and so \( \tilde{A}_{n-1} \) is a split extension of \( S_n \) by \( A \) where \( n \geq 3 \).

PROOF. In Proposition 1, we let \( a_j x_1 = b \). Then \( E \) has the following generators:
\( x_1, x_2, \ldots, x_{n-1}, b \). Relations of \( E \) are:

Relations of \( S_n \),
\[
b^2 = e, \quad (2.10)
\]
\[
(b, x_2) \quad (2.11)
\]
\[
[\text{bx}_1, x_i] \text{ for } 3 \leq i \leq n-1 \quad (2.12)
\]
\[
(x_2 bx_1)^2 = (bx_1 x_2)^2. \quad (2.13)
\]

We change relation (2.13) to the form
\[
(b, x_1 x_2 x_1). \quad (2.14)
\]

We change relation (2.12) to \([b, x_i] \text{ for } 3 \leq i \leq n-1 \) \( (2.15)\)

We let \( c = \Delta_{n-1}^{-1} b \Delta_{n-1} \). Then \( c^2 = e \).

Using relation (2.11) and Lemma 1 (i), we get \((c, x_1)\).
\[
x_{n-1} c x_{n-1} = \Delta_{n-1}^{-1} b \Delta_{n-1}. \quad (2.16)
\]
Using Lemma 1 (ii) and (v) and (2.15)
\[ cx_{n-1}c = \Delta_{n-1}^{-2} b \Delta_{n-1} \]
Using Lemma 1 (vi)
\[ cx_{n-1}c = \Delta_{n-2}^{-1} b \Delta_{n-2}^2 = x_{n-1} c x_{n-1} \]
Therefore \((c, x_{n-1})\).
Using Lemma 1 (i) and (2.15) we get
\([c, x_i]\) for \(2 < i < n-1\).
Thus \(E\) has the following presentation
\[ E = \langle x_1, \ldots, x_{n-1}, c | x_i^2 = e \quad \text{for} \quad 1 \leq i \leq n-1, \]
\[ c^2 = e, \]
\[ (x_i, x_{i+1}) \quad \text{for} \quad 1 \leq i \leq n-2 \]
\[ [x_i, x_k] \quad \text{for} \quad 1 \leq i < k-1 < n-1, \]
\[ (x_{n-1}, c), (x_1, c), \]
\[ [x_i, c] \quad \text{for} \quad 2 \leq i \leq n-1. \]
Let \(c = x_n\). Then it is clear that \(E\) is the same as \(\tilde{A}_{n-1}\) and the theorem is proved.

**Remark 1.** We notice the special cases \(\tilde{A}_0 = S_2 = Z_2\), \(\tilde{A}_1 = S_3\), \(\tilde{A}_2 = \Delta(3, 3, 3)\) the triangle group \(\Delta(3, 3, 3)\) [6].

**Remark 2.** We used the Reidemeister-Schreier process to find \(A = \ker \theta\) for \(n = 3, 4\). From the computations involved we found the action of \(S_n\) on \(A\). For \(n > 5\), we guessed that \(A = Z^{x(n-1)}\) and the action is a generalization for the case when \(n = 3, 4\). We then proved this guess by the method in [6].

3. THE DERIVED SERIES OF \(\tilde{A}_{n-1}\):

We prove in this section the following theorem:

**Theorem 2.** The group \(\tilde{A}_3\) is soluble of derived length 3, \(\tilde{A}_4\) is soluble of derived length 4. For \(n > 4\), the second derived group \(\tilde{A}_{n-1}''\) coincides with the first \(\tilde{A}_{n-1}'\) and so \(\tilde{A}_{n-1}''\) is not soluble for \(n > 4\).

To prove the theorem we consider the derived series of \(A_{n-1}\). We notice that \(\frac{A_{n-1}}{A_{n-1}'} = \langle y_1 | y_1^2 \rangle\). Hence \((e, y_1)\) is a transversal for \(\tilde{A}_{n-1}'\) in \(\tilde{A}_{n-1}\). Using the Reidemeister-Shreier process we find the following presentation for \(\tilde{A}_{n-1}'\):
\[ \tilde{A}_{n-1}' = \langle b_1, b_2, \ldots, b_{n-1} | b_i^2 = b_{i+1}^3 = e, \quad \text{if} \quad 1 \leq i \leq n-2, \]
\[ (b_ib_{i+1}^{-1})^3 = e, \quad \text{if} \quad 1 \leq i < n-2, \]
\[ (b_ib_j^{-1})^2 = e, \quad \text{if} \quad 1 \leq i < j-1 < n-1. \]
We now consider the following cases:

i) If \( n = 3 \), \( \frac{A_2}{\langle b_1, b_2 \rangle} = \langle b_1^3, b_2^3 \rangle = \langle [b_1, b_2] \rangle = e \).

Using the Reidemeister-Schreier process we find that \( A_3 = z \times z \).

Therefore \( A_2 = 1 \) is soluble of derived length 3.

ii) If \( n = 4 \), \( \frac{A_3}{\langle b_1^2 \rangle} = \langle b_1^3 \rangle = e \). We use the Reidemeister-Schreier process to find the following presentation for \( \frac{A_3}{A_3''} \):

\[
\frac{A_3}{A_3''} = \langle x, y, z, t | x^2 = y^2 = z^2 = t^2 = [x, z] = [y, t] = e \rangle
\]

\[
\begin{align*}
\frac{A_3}{A_3''} &= \langle x, y, z, t | x^2 = y^2 = z^2 = t^2 = [x, z] = [x, t] = [y, x] = [y, t] = [z, t] = e \rangle \\
&= \langle x, y | [x, y] = [x, z] = [x, t] = [y, x] = [y, t] = [z, t] = e \rangle.
\end{align*}
\]

We use the Reidemeister-Schreier process to find that \( \frac{A_3''}{A_3'''} = z \times z \). Therefore \( A_3 = 1 \) is soluble at derived length 4.

iii) If \( n > 4 \), \( \frac{A_{n-1}}{A_{n-1}'} \) is trivial. So the second derived group \( \frac{A_{n-1}''}{A_{n-1}'''} \) coincides with first derived group \( A_{n-1}'' \). Hence \( A_{n-1}'' \) is not soluble for \( n > 4 \).

4. THE CENTER OF \( A_{n-1}'' \)

We prove in this section that the center of \( A_{n-1}'' \) is trivial for \( n \geq 3 \).

**LEMMA 2.** The identity of \( A \) is the only element fixed by \( S_n \).

**PROOF.** We let \( w \) be an element of \( A \). We can write \( w \) in the form

\[
a_1 a_2 \ldots a_{n-1}
\]

where \( a_j \in A \) for \( 1 \leq j \leq n-1 \). Let \( w^i = w \) for \( 1 \leq i \leq n-1 \).

We therefore get the equation

\[
\left[ \begin{array}{ccc}
\alpha_1 & a_2 & \ldots & a_{n-1} \\
\beta_1 & m_2 & \ldots & m_{n-1} \\
\end{array} \right] = a_1 a_2 \ldots a_{n-1}
\]

for \( 1 \leq i \leq n-1 \).

Using the action of \( S_n \) on \( A \) [in Section 2] equation (4.1) for \( i = 1 \) implies

\[
2m_1 + m_2 + \ldots + m_{n-1} = e.
\]

Since \( A \) is free abelian this equation gives

\[
2m_1 + m_2 + \ldots + m_{n-1} = 0.
\]

Using the action of \( S_n \) on \( A \), equation (4.1) for \( 2 \leq i \leq n-1 \) implies

\[
\frac{m_i - m_{i-1}}{a_{i-1}} = \frac{m_i - m_{i-1}}{a_{i-1}}.
\]

Since \( A \) is free abelian this gives

\[
m_i - m_{i-1} = 0 \text{ for } 2 \leq i \leq n-1.
\]

From (4.2) and (4.3) we get \( m_1 = m_2 = \ldots = m_{n-1} = 0 \). Therefore \( w = e \) as required.
THEOREM 3. The center of \( \tilde{A}_{n-1} \) is trivial for \( n \geq 3 \).

PROOF. We know that

\[
1 \rightarrow A \rightarrow \tilde{A}_{n-1} \rightarrow S_n \rightarrow 1.
\]

We let \( x \in Z(\tilde{A}_{n-1}) \) so \( x = as \) where \( a \in A \) and \( s \in S_n \). We let \( x_1 = a_1s_1 \) be a typical element of \( \tilde{A}_{n-1} \). Hence \( xx_1 = x_1x \) implies \( as_1s_1 = a_1s_1as \). Applying the epimorphism \( \phi \) we get \( \phi(s)\phi(s_1) = \phi(s_1)\phi(s) \) and so \( \phi(s) \in Z(S_n) = \{e\} \). Hence \( s \in \ker \phi = A \cap S_n = e \). Therefore \( x = e \) commutes elementwise with \( S_n \). Using Lemma 3, \( a = e \) and so \( Z(\tilde{A}_{n-1}) = \{e\} \).

REMARK 3. From Remark 1 we notice that \( Z(A_0) = Z \) and \( Z(\tilde{A}_1) = Z(S_3) = \{e\} \).

REMARK 4. We notice that \( \tilde{A}_{n-1} \cong S_n \) from Theorem 3. Since \( S_3 \) and \( S_4 \) are soluble of length 3 and 4 respectively, we get that \( \tilde{A}_2 \) and \( \tilde{A}_3 \) are soluble of length 3 and 4 respectively. \( S_n \) is not soluble for \( n > 4 \) and \( A \) is soluble, it follows that \( \tilde{A}_{n-1} \) is not soluble for \( n > 4 \).

REMARK 5. One way to view \( \tilde{A}_{n-1} \) is as a subgroup of the wreath product \( Z S_n \) defined as follows: Let \( Z^{\times n} \) be the free abelian group with base \( P_0, \ldots, P_{n-1} \) on which \( S_n \) acts by permuting the basis, \( x_i = (i-1, i) \), exchanges \( P_{i-1} \) and \( P_i \) and fixes the others. The subgroup \( \langle P_0 \ldots P_{n-1} | k_{n-1} = 0 \rangle = H \) is \( S_n \)-invariant, and has basis \( \{a_i = P_i - P_0 | 1 \leq i \leq n-1, \} \) and \( \tilde{A}_{n-1} \) is just this split extension of \( S_n \) by \( H \). Therefore \( \tilde{A}_{n-1} \) is the subgroup of the natural wreath product of \( Z S_n \) consisting of those elements in which the component from the base group has exponent sum zero.

REMARK 6. The motivation behind studying this group \( \tilde{A}_{n-1} \) was to get some information about the circular braid group \( B_n \) [7]. We see that \( \tilde{A}_{n-1} \) is the Coxeter group corresponding to the Artin group \( B_n \). Consider the diagram

\[
\begin{align*}
1 & \rightarrow Y \rightarrow F \rightarrow Z^{\times (n-1)} \rightarrow 1 \\
1 & \rightarrow X \rightarrow B_n \rightarrow \tilde{A}_{n-1} \rightarrow 1 \\
U_n & \rightarrow B_n \rightarrow S_n \rightarrow 1 \\
& \rightarrow 1
\end{align*}
\]

Here \( B_n \) is Artin's braid group [6], \( U_n \) the unpermuted braid group, \( F \) a free group of countably infinite rank [7] and \( Z^{\times (n-1)} \) as described in this paper. Knowing \( Z^{\times (n-1)} \) did not help us to describe the structure of \( B_n \) which was described in a different way [7]. We are still unable to find the groups \( X \) and \( Y \).

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