DiffEomorphism Groups of Connected Sum of a Product of
Spheres and Classification of Manifolds

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ABSTRACT. In [1] and [2] a classification of a manifold $M$ of the type $(n,p,l)$ was
given where $H_p(M) = H_{n-p}(M) \cong \mathbb{Z}$ is the only non-trivial homology groups. In this
paper we give a complete classification of manifolds of the type $(n,p,2)$ and we
extend the result to manifolds of type $(n,p,r)$ where $r$ is any positive integer
and $p = 3,5,6,7 \pmod{8}$.

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0. INTRODUCTION.

In [1] Edward C. Turner worked on a classification of a manifold $M$ of the type $(n,p,r)$ where this means that $M$ is simply connected smooth $n$-manifold and
$H_p(M) = H_{n-p}(M) \cong \mathbb{Z}$ the only non-trivial homology groups except for the top and
bottom groups. He gave a classification of such manifolds for the case $r = 1$ and
$p = 3,5,6,7 \pmod{8}$. So Turner gave a classification of $M$ of type $(n,p,1)$ and
$p = 3,5,6,7 \pmod{8}$. In [2] Hajime Sato independently obtained similar results for
$M$ of the type $(n,p,1)$. The question which naturally follows is: Suppose $r = 2,3,4$
and so on, what is the classification of such $M$? i.e., what is the classification
of $M$ of the type $(n,p,2)$, $(n,p,3)$ and so on? In this paper we will study
manifolds for the type $(n,p,2)$ and give its complete classification and then gen-
eralize the result to manifolds $M$ of the type $(n,p,r)$ where $r$ is an integer
and $p = 3,5,6,7 \pmod{8}$.

In §1 we prove the following

THEOREM 1.1 Let $M$ be an $n$-dimensional oriented, closed, simply connected
manifold of the type $(n,p,2)$ with $p = 3,5,6,7 \pmod{8}$. Then $M$ is diffeomorphic
to $S^p \times D^{q+1} \# S^p \times D^{q+1} \cup S^p \times D^{q+1} \# S^q \times D^{p+1}$ where $n = p+q+1$,
$h$ means connected
sum along the boundary as defined by Milnor and Karvair [3] and $h : S^p \times S^q \# S^p \times S^q \to
\to S^p \times S^q \# S^p \times S^q$ is a diffeomorphism.

In §2 we compute the group $\pi_0 \text{Diff}(S^p \times S^q \# S^p \times S^q)$ of pseudo-diffeotopy
classes of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$.
Let $GL(2, \mathbb{Z})$ denote the set of $2 \times 2$ unimodular matrices and $H$ the subgroup of $GL(2, \mathbb{Z})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - cd = 0 \mod 2$ and $\mathbb{Z}_4$ the subgroup of $GL(2, \mathbb{Z})$ of order 4 generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We will adopt the notation $M_{p, q} = \text{Diff}(S^p \times S^q \# S^p \times S^q)$ and $M^+_{p, q}$ the subgroup of $M_{p, q}$ consisting of diffeomorphisms which induce identity map on all homology groups. We will then prove the following

**Theorem 2.1**

(i) If $p + q$ is even, then

$$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \text{ if } p \text{ is even, } q \text{ is even}$$

$$GL(2, \mathbb{Z}) \oplus GL(2, \mathbb{Z}) \text{ if } p, q = 1, 3, 7$$

$$H \oplus H \text{ if } p, q \text{ odd but } \not\equiv 1, 3, 7$$

$$GL(2, \mathbb{Z}) \cup \mathbb{Z}_4 \text{ if } p \text{ is even, } q \text{ is odd but } \not\equiv 1, 3, 7$$

(ii) If $p + q$ is odd then

$$\mathbb{Z}_4 \oplus H \text{ if } p \text{ is even, } q \text{ odd but } \not\equiv 1, 3, 7$$

$$\mathbb{Z}_4 \oplus GL(2, \mathbb{Z}) \text{ if } p \text{ is even and } q = 1, 3, 7$$

We will further prove the following.

**Theorem 2.15** If $p < q$ and $p = 3, 5, 6, 7 \mod 8$ the order of the group $\pi_0(M^+_{p, q})$ is twice the order of the group $\pi_q(SO(p+1)) \circ \mathbb{Z}^{p+q+1}$.

In §3 we apply the result in §2 to prove the following

**Theorem 3.7** Let $M$ be an $n$-dimensional, smooth, closed, oriented manifold such that $n = p+q+1$ and

$$H_i(M) = \begin{cases} \mathbb{Z} & i = 0, n \\ \mathbb{Z} \oplus \mathbb{Z} & i = p, q+1 \\ 0 & \text{elsewhere} \end{cases}$$

then if $p = 3, 5, 6, 7 \mod 8$ the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to twice the order of the group $\pi_q(SO(p+1)) \circ \mathbb{Z}^{p+q+1}$. With induction hypothesis and technique used in §1 and §2, one can prove the following

**Theorem 3.8** If $M$ is a smooth, closed simply connected manifold of type $(n, p, r)$ where $n = p+q+1$ and $p = 3, 5, 6, 7 \mod 8$, then the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to

$$r \times \text{order of } \pi_q(SO(p+1)) \circ \mathbb{Z}^{p+q+1}$$

1. **MANIFOLDS OF TYPE $(n, p, r)$**

**Definition:** Let $M$ be a closed, simply connected $n$-manifold. $M$ is said to be of type $(n, p, r)$ if

$$H_i(M) = \begin{cases} \mathbb{Z} & i = 0, n \\ \mathbb{Z}^r & i = p, q+1 \\ 0 & \text{elsewhere} \end{cases}$$

where $n = p+q+1$

We recall from Milnor and Kervaire [3]

**Definition:** Let $M_1$ and $M_2$ be $(p+q+1)$-manifolds with boundary and $H^{p+q+1}$
be half-disc, i.e.,

$$H^{p+q+1} = \left\{ (x = x_1, x_2, \ldots, x_{p+q+1} \mid x_1 \leq 1, x_1 \geq 0) \right\}$$

Let $D^{p+q}$ be the subset of $H^{p+q+1}$ for which $x_1 = 0$. We can choose embeddings

$$i_\alpha : (H^{p+q+1}, D^{p+q}) \rightarrow (M, \partial M), \quad \alpha = 1, 2$$

so that $i_2^{-1} i_1$ reverses orientation. We then form the sum $(M_1 - i_1(0)) \cup (M_2 - i_2(0))$ by identifying $i_1(tu)$ with $i_2((1-t)u)$ for $0 < t < 1, u \in S^{p+q} \cap H^{p+q+1}$. This sum is called the connected sum along the boundary and will be denoted by $M_1 \# M_2$.

**REMARK:** (1) Notice that the boundary of $M_1 \# M_2$ is $\partial M_1 \# \partial M_2$.

(2) $M_1 \# M_2$ has the homotopy type of $M_1 \vee M_2$ : the union with a single point in common.

**THEOREM 1.1** If $M$ is a smooth manifold of type $(n, p, 2)$ where $n = p+q+1$ and $p = 3, 5, 6, 7$ (mod 8) then there exists a diffeomorphism

$$h : S^p \times S^q \# S^p \times S^q \rightarrow S^p \times S^q \# S^p \times S^q$$

which induce identity on homology such that $M$ is diffeomorphic to

$$S^p \times D^{q+1} \# S^p \times D^{q+1}$$

**PROOF:** Let $[M, \lambda_1, \lambda_2]$ be a manifold of type $(n, p, 2)$ and $\lambda_1, \lambda_2$ represent the generators of the first and second summands of $H_p(M) = \mathbb{Z} \oplus \mathbb{Z}$. We can choose embeddings $\varphi_1 : S^p \rightarrow M$ so as to represent the homology class $\lambda_1, i = 1, 2$. Since $p < q$, two homotopic embeddings are isotopic. Let $\alpha_1 \in \pi_{p+q}(SO(q+1))$ be the characteristic class of the embedded sphere $S^p$, since $p = 3, 5, 6, 7$ (mod 8), the normal bundle of the embedded sphere is trivial. It follows that $\varphi_1$ extends to an embedding $\varphi_1' : S^p \times D^{q+1} \rightarrow M$ such that its homology class is $\lambda_1$. Then we can form a connected sum along the boundary of the two embedded copies of $S^p \times D^{q+1}$ to get $S^p \times D^{q+1} \# S^p \times D^{q+1}$. We then have an embedding $i : S^p \times D^{q+1} \# S^p \times D^{q+1} \rightarrow M$ such that $i_*[S^p] = \lambda_1 + \lambda_2 \in H_p(M)$. Notice that the boundary of $S^p \times D^{q+1} \# S^p \times D^{q+1}$ is $S^p \times S^q \# S^p \times S^q$ and since $S^p \times D^{q+1} \# S^p \times D^{q+1}$ has the homotopy type of $S^p \times D^{q+1} \# S^p \times D^{q+1}$ then it is easy to see that

$$H_*(S^p \times D^{q+1} \# S^p \times D^{q+1}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\
\mathbb{Z} \oplus \mathbb{Z} & \text{for } i = p \end{cases}$$

It is also easy to see that

$$H_*(M-\text{Int}(S^p \times D^{q+1} \# S^p \times D^{q+1})) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\
\mathbb{Z} \oplus \mathbb{Z} & \text{for } i = p \end{cases}$$

Now since $S^p \times D^{q+1}$ is a trivial disc bundle over $S^p$ then it has cross sections; hence, there exists orientation reversing diffeomorphism of $S^p \times D^{q+1} \# S^p \times D^{q+1}$ onto itself. Thus there exists an orientation reversing embedding

$$j : S^p \times D^{q+1} \# S^p \times D^{q+1} \rightarrow M-\text{Int}(S^p \times D^{q+1} \# S^p \times D^{q+1})$$
such that \( j_{p} = \lambda_{1} + \lambda_{2} \) and in fact this embedding is a homotopy equivalence. It follows by [4, Thm. 4.1] that \( S^{p} \times S^{q+1} \# S^{p} \times D^{q+1} \) is diffeomorphic to
\[
\partial M - \text{Int}(S^{p} \times D^{q+1} \# S^{p} \times D^{q+1}) .
\]
Consequently, it follows that \( M \) is diffeomorphic to
\[
S^{p} \times D^{q+1} \# S^{p} \times D^{q+1} \cup S^{p} \times D^{q+1} \# S^{p} \times D^{q+1}
\]
for an orientation preserving diffeomorphism
\[
h: S^{p} \times S^{q} \# S^{p} \times S^{q} \to S^{p} \times S^{q} \# S^{p} \times S^{q} .
\]
From the embeddings in the proof, it is clear that \( h \) induce identity on homology.

2. THE GROUP \( \tilde{\pi}_{0}(\text{Diff}(S^{p} \times S^{q} \# S^{p} \times S^{q})) \)

For convenience, we adopt the notation \( M_{p, q} = \text{Diff}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \) and \( \tilde{\pi}_{0}(M_{p, q}) \) the subset of \( M_{p, q} \) consisting of diffeomorphisms of \( S^{p} \times S^{q} \# S^{p} \times S^{q} \) which induce identity on all homology groups.

DEFINITION: Let \( M \) be an oriented smooth manifold. \( \text{Diff}(M) \) is the group of orientation preserving diffeomorphisms of \( M \). Let \( f, g \in \text{Diff}(M) \), \( f \) and \( g \) are said to be pseudo-diffeotopic if there exists a diffeomorphism \( H \) of \( M \times I \) such that \( H(x, 0) = (f(x), 0) \) and \( H(x, 1) = (g(x), 1) \) for all \( x \in M \). The pseudo-diffeotopy class of diffeomorphisms of \( M \) is denoted by \( \tilde{\pi}_{0}(\text{Diff}(M)) \). We wish to compute
\[
\tilde{\pi}_{0}(M_{p, q}) \text{ for } p < q .
\]
If \( f, \in M_{p, q} \), then \( f \) induces an automorphism
\[
f_{\ast}: H_{\ast}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \to H_{\ast}(S^{p} \times S^{q} \# S^{p} \times S^{q})
\]
of homology groups of \( S^{p} \times S^{q} \# S^{p} \times S^{q} \). Since pseudo-diffeotopic diffeomorphisms induce equal automorphism on homology then we have a well-defined homomorphism
\[
\hat{\psi}: \tilde{\pi}_{0}(M_{p, q}) \to \text{Auto}(H_{\ast}(S^{p} \times S^{q} \# S^{p} \times S^{q}))
\]
where \( \text{Auto}(H_{\ast}(S^{p} \times S^{q} \# S^{p} \times S^{q})) \) denotes the group of dimension preserving automorphisms of \( H_{\ast}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \).

THEOREM 2.1 (i) If \( p+q \) is even then
\[
\#(\tilde{\pi}_{0})(M_{p, q}) = \begin{cases} Z_{4} \oplus Z_{4} & \text{if } p, q \text{ are even} \\ \text{GL}(2, Z) \oplus \text{GL}(2, Z) & \text{if } p, q \text{ are } 1, 3, 7 \\ H \circ H & \text{if } p, q \text{ are odd but } \neq 1, 3, 7 \\ \text{GL}(2, Z) \oplus H & \text{if } p = 1, 3, 7 \text{ and } q \text{ is odd but } \neq 1, 3, 7 \end{cases}
\]
The following propositions give the proof of Theorem 2.1.

PROPOSITION 2.1 If \( p+q \) is even, \( p \) is even, then
\[
\#(\tilde{\pi}_{0})(M_{p, q}) = Z_{4} \oplus Z_{4} .
\]

PROOF: Since \( p+q \) is even and \( p \) is even then \( q \) must also be even. We have
\[
H_{i}(S^{p} \times S^{q} \# S^{p} \times S^{q}) = \begin{cases} Z & \text{if } i = 0, p+q \\ Z \oplus Z & \text{if } i = p \text{ or } q \\ 0 & \text{elsewhere} \end{cases}
\]
Generators of \( H_{0}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \) and \( H_{p+q}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \) are mapped to the same generators but \( H_{p}(S^{p} \times S^{q} \# S^{p} \times S^{q}) = Z \oplus Z \). If \( f \in M_{p, q} \), we shall denote by \( \hat{\psi}(f)_{p} \) the automorphism \( f_{\ast}: H_{p}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \to H_{p}(S^{p} \times S^{q} \# S^{p} \times S^{q}) \) induced by the image \( f \) under \( \hat{\psi} \) in dimension \( p \). Then \( \hat{\psi}(f)_{p} = f_{\ast}: Z \oplus Z \to Z \oplus Z \) is the induced
automorphism. If \( e_1, e_2 \) are the generators of the first and second summand of 
\( H_p(S^p \times S^q \# S^p \times S^q) \) if \( \circ \) denotes the intersection then \( e_1 \circ e_1 = 0 \), \( e_2 \circ e_2 = 0 \), 
\( e_1 \circ e_2 = 1 \) and \( e_2 \circ e_1 = -1 \). Let \( \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL(2, Z) \), if \( \xi(f)_p \) takes \( e_1, e_2 \) to 
\( e'_1, e'_2 \) respectively then \( e'_1 = a_1 e_1 + a_2 e_2 \) and \( e'_2 = a_3 e_1 + a_4 e_2 \) then 
\[
\begin{align*}
e'_1 \circ e'_1 &= (a_1 e_1 + a_2 e_2) \cdot (a_1 e_1 + a_2 e_2) \\
&= a_1 a_1 e_1 \cdot e_1 + a_2 a_1 e_2 \cdot e_2 + a_2 a_2 e_2 \cdot e_1 + a_1 a_2 e_2 \cdot e_2 \\
&= a_1 a_1 e_1 \cdot e_1 + a_2 a_2 e_2 \cdot e_2 = a_1^2 - a_2^2 = 0.
\end{align*}
\]
Similarly \( e'_2 \circ e'_2 = 0 \) but 
\[
\begin{align*}
e'_1 \circ e'_1 &= a_1 a_1 e_1 \cdot e_1 + a_2 a_1 e_2 \cdot e_2 + a_2 a_2 e_2 \cdot e_1 + a_1 a_2 e_2 \cdot e_2 \\
&= a_1 a_1 e_1 \cdot e_1 + a_2 a_2 e_2 \cdot e_2 = a_3 a_4 - a_3 a_4 = 1 \text{ since } GL(2, Z) \text{ is unimodular.}
\end{align*}
\]
the for \( p \) even \( \xi(f)_p \) is an element of a subgroup of \( GL(2, Z) \) generated by 
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
This subgroup has elements \( \{ (\pm 1 0, 0 \pm 1) \} \approx Z_4 \). Hence \( \xi(f)_p \in Z_4 \). 
Similarly for \( i = q \) \( \xi(f)_q \in Z_4 \), it then follows that 
\[
\xi(\pi_0(M, p, q)) = Z_4 \circ Z_4.
\]
We now show that \( Z_4 \circ Z_4 \subset \xi(\pi_0(M, p, q)) \). We need to show that the generators 
of \( Z_4 \circ Z_4 \) can be realized as the image of \( \xi \). We shall adopt the notation 
\( (s^p \times s^q)_1 \# (s^p \times s^q)_2 \) where the subscripts 1 and 2 denote the first and second summands 
of \( s^p \times s^q \# s^p \times s^q \) and let \( R_p \) and \( R_q \) be reflections of \( s^p \) and \( s^q \) respectively. 
If \( (x_1, y_1) \in (s^p \times s^q)_1 \) and \( (x_2, y_2) \in (s^p \times s^q)_2 \), we define \( f \in M_{p, q} \) 
\[
\begin{align*}
f(x_1, y_1) &= (R_p(x_2), R_q(y_2)) \\
f(x_2, y_2) &= (x_1, y_1)
\end{align*}
\]
In other words \( f((x_1, y_1)(x_2, y_2)) = ((R_p(x_2), R_q(y_2)), (x_1, y_1)) \) 
\( (x_1, y_1) \in (s^p \times s^q)_1 \) and \( (x_2, y_2) \in (s^p \times s^q)_2 \).

For \( \xi(f)_p \in Aut H_p(M, p, q) \), if \( e_1, e_2 \) are the generators of the first and 
second summands of \( H_p(s^p \times s^q \# s^p \times s^q) = Z \circ Z \) since \( f \) takes \( x_1 \) to \( R_p(x_2) \) 
and \( f \) takes \( x_2 \) to \( x_1 \), then it is easily seen that \( \xi(f)_p(e_1) = -e_2 \) and 
\( \xi(f)_p(e_2) = e_1 \). Hence \( e'_1 = -e_2 \) and \( e'_2 = e_1 \) so an \( e'_0 e'_1 = -e_2 \circ e_2 = 0 \), 
\( e'_2 e'_1 = e_0 e_1 = 0 \), \( e'_0 e'_2 = -e_2 \circ e_1 = 1 \) and \( e'_0 e'_1 = e_0 \circ e_2 = -1 \) . Hence \( \xi \) 
maps \( f \) in dimension \( p \) to \( (0 1) \) which generates \( Z_4 \). Similar argument shows 
that \( \xi \) maps \( f \) in dimension \( q \) to \( (1 0 -1 -1) \) which generates \( Z_4 \). Then \( \xi \) maps 
onto \( Z_4 \circ Z_4 \) hence the proof.

**Proposition 2.2** If \( p+q \) is even but \( p, q = 1, 3, 7 \) then 
\[
\xi(\pi_0(M, p, q)) = GL(2, Z) \circ GL(2, Z)
\]

**Proof:** From [5, Appendix B] and [6] one sees that \( GL(2, Z) \) is generated by 
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
Since \( p, q = 1, 3, 7 \) it follows by [7, §1] that there exist maps \( f: s^p \rightarrow SO(p+1) \) and 
\( g: s^q \rightarrow SO(q+1) \) such that \( f \) and \( g \) have index +1.
We then define \( h \in M_{p,q} \)

\[
\begin{align*}
  h(x_1, y_1) &= (x_1, y_1) \\
  h(x_2, y_2) &= (f(x_1) \cdot x_2, g(y_1) \cdot y_2)
\end{align*}
\]

i.e., \( h((x_1, y_1), (x_2, y_2)) = ((x_1, y_1), (f(x_1) \cdot x_2, g(y_1) \cdot y_2)) \)

Since \( f \) has index +1 and \( h \) takes \( x_1 \) to \( x_1 \) and \( x_2 \) to \( f(x_1) \cdot x_2 \) then it follows by an easy application of \([7, \text{Prop. 1.2}] \) or \([6, \text{Prop. 2.3}] \) that \( \hat{\gamma}(h)_p \) is \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) also since \( g \) has index +1 and \( h \) takes \( y_1 \) to \( y_1 \) and \( y_2 \) to \( g(y_1) \cdot y_2 \)

then \( \hat{\gamma}(h)_q \) is \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Hence \( \hat{\gamma} \) maps \( h \) to \( \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \). We now define \( \alpha \in M_{p,q} \) by

\[
\begin{align*}
  \alpha(x_1, y_1) &= (R_p(x_2), R_q(y_2)) \\
  \alpha(x_2, y_2) &= (x_1, y_1)
\end{align*}
\]

i.e., \( \alpha((x_1, y_1), (x_2, y_2)) = ((R_p(x_2), R_q(y_2)), (x_1, y_1)) \)

Since \( \alpha \) takes \( x_1 \) to \( R_p(x_2) \) and \( x_2 \) to \( x_1 \) it follows from Proposition 2.1 that \( \hat{\gamma}(\alpha)_p \) is \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and by similar reasoning \( \hat{\gamma}(\alpha)_q \) is \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). This means that \( \hat{\gamma} \) maps \( \alpha \) to \( \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \). Since \( GL(2, Z) \) is generated by \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) then it follows that for \( p, q = 1, 3, 7 \)

\[
\hat{\gamma}(\alpha((M_{p,q})) \approx GL(2, Z) \otimes GL(2, Z).
\]

**Proposition 2.3** If \( p+q \) is even but \( p \) and \( q \) are odd but \( p, q \neq 1, 3, 7 \), then \( \hat{\gamma}(\alpha((M_{p,q})) \approx H \otimes H \).

**Proof:** By using Proposition 2.1 and \([8, \text{Lemma 5}] \) it is enough to produce a diffeomorphism in \( M_{p,q} \) whose image under \( \hat{\gamma} \) is \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) in each of the dimensions \( p \) and \( q \). This is because \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) generate \( H \). However, \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is trivially the image under \( \hat{\gamma} \) of identity map and reflections on each coordinate while \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is by Proposition 2.1 the image under \( \hat{\gamma} \) of an element of \( M_{p,q} \). However, there exists a map \( \alpha : S^p \rightarrow SO(p+1) \) of index 2 by \([8] \) also is a map \( \beta : S^q \rightarrow SO(q+1) \) of index 2 and then we can define \( f \in M_{p,q} \) thus.

\[
\begin{align*}
  f(x_1, y_1) &= (x_1, y_1) \\
  f(x_2, y_2) &= (\alpha(x_1) \cdot x_2, \beta(y_1) \cdot y_2)
\end{align*}
\]

i.e., \( f((x_1, y_1), (x_2, y_2)) = ((x_1, y_1), (\alpha(x_1) \cdot x_2, \beta(y_1) \cdot y_2)) \).

It easily follows that since \( f \) takes \( x_1 \) to \( x_1 \) and takes \( x_2 \) to \( \alpha(x_1) \cdot x_2 \) with \( \alpha \) having index 2 then it follows by applying \([7, \text{Lemma 5}] \) that \( \hat{\gamma}(f)_p \) is \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \). Similar argument shows that \( \hat{\gamma}(f)_q \) is \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \); hence \( \hat{\gamma}(\alpha((M_{p,q})) \approx H \otimes H \).

**Proposition 2.4** If \( p+q \) is even, \( p=1, 3, 7 \) but \( q \) is odd and \( \neq 1, 3, 7 \) then \( \hat{\gamma}(\alpha((M_{p,q})) = GL(2, Z) \otimes H \).

**Proof:** \( \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\} \) generates \( GL(2, Z) \) while \( \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\} \) generates \( H \), since \( q \neq 1, 3, 7 \) and by \([8] \) there exists \( \alpha : S^q \rightarrow SO(q+1) \) of index 2. If \( R_p \) is reflection of \( S^p \) then we define \( h \in M_{p,q} \).
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\[ h(x_1, y_1) = (R_p(x_2), y_1) \quad (x_1, y_1) \in (S^p \times S^q)_1 \]
\[ h(x_2, y_2) = (x_1, \alpha(y_1), y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2 \]

Since \( h \) takes \( x_1 \) to \( R_p(x_2) \) and takes \( x_2 \) to \( x_1 \) it follows by Proposition 2.1 that \( \hat{h}(p)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \); similarly \( h \) takes \( y_1 \) to \( y \), and \( y_2 \) to \( \alpha(y_1) \cdot y_2 \) and since \( \alpha \) has index 2, it follows that \( \hat{h}(q)_q = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \).

Now if \( R_q \) is a reflection on \( S^q \) and \( \beta : S^p \to SO_{p+1} \) is of index +1 then we define \( f \in \mathcal{M}_{p,q} \)

\[ f(x_1, y_1) = (x_1, R_q(y_2)) \quad (x_1, y_1) \in (S^p \times S^q)_1 \]
\[ f(x_2, y_2) = (\beta(x_1), x_2, y_1) \quad (x_2, y_2) \in (S^p \times S^q)_2 \]

then it is easy to see that \( \hat{f}(p)_p = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \hat{f}(q)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) so the image of \( h \) under \( \hat{f} \) is \( \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\} \) and the image of \( f \) under \( \hat{f} \) is \( \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \) which generates \( H \) and \( \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \) which generates \( GL(2,\mathbb{Z}) \) then it follows that \( \hat{f}(\mathcal{M}_{p,q}) \cong GL(2,\mathbb{Z}) \) \( \oplus H \). Hence the proof.

REMARK. For \( p \) odd but \( \neq 1,3,7 \) and \( q=1,3,7 \), we have the same result as above using the same method but since by assumption \( p < q \) only one dimension (consequently one manifold) comes in here, viz \( p=5 \), \( q=7 \), i.e., \( S^5 \times S^7 \# S^5 \times S^7 \).

Combination of Propositions 2.1, 2.2, 2.3, and 2.4 proves Theorem 2.1(i).

PROPOSITION 2.5 Suppose \( p+q \) is odd and \( p \) is even and \( q \) odd \( \neq 1,3,7 \) then
\[ \hat{f}(\mathcal{M}_{p,q}) \cong Z_4 \oplus H \]
PROOF: Again since \( q = 1, 3, 7 \) by [6, Prop. 2.4] there exists a map \( \alpha: S^q \to SO(q+1) \) of index 1. Since \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) generates \( Z_4 \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) generate \( GL(2, Z) \) we define elements of \( M_{p, q} \) that are mapped onto these generators. Let \( h \in M_{p, q} \) be defined thus

\[
\begin{align*}
&h(x_1, y_1) = (R_p(x_2), y_1) \quad \text{where} \quad (x_1, y_1) \in (S^p \times S^q)_1 \\
&h(x_2, y_2) = (x_1, \alpha(y_1), y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2
\end{align*}
\]

i.e., \( h(x_1, y_1), (x_2, y_2) = ((R_p(x_2), y_1), (x_1, \alpha(y_1), y_2)) \) where \( R_p \) is the reflection of \( S^p \) Then it is easy to see that \( \hat{\psi}(h) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) while \( \hat{\psi}(h) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \) Also one can define \( f \in M_{p, q} \) as

\[
\begin{align*}
&f(x_1, y_1) = (x_1, R_q(x_2)) \quad \text{where} \quad (x_1, y_1) \in (S^p \times S^q)_1, (x_2, y_2) \in (S^p \times S^q)_2 \\
&f(x_2, y_2) = (x_2, y_1)
\end{align*}
\]

i.e., \( f((x_1, y_1), (x_2, y_2)) = ((x_1, R_q(x_2)), (x_2, y_1)) \) where \( R_q \) is a reflection of \( S^q \) and so it is easily seen that \( \hat{\psi}(f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) while \( \hat{\psi}(f) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) so \( h \) is mapped by \( \hat{\psi} \) to \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) while \( f \) is mapped by \( \hat{\psi} \) to \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and since these sets of matrices generate \( GL(2, Z) \) and \( Z_4 \) respectively then \( \hat{\psi}(M_{p, q}) \approx Z_4 \otimes GL(2, Z) \). Combining Propositions 2.5 and 2.6, we obtain Theorem 2.1 (ii).

REMARK. If \( p \) is odd but \( \neq 1, 3, 7 \) and \( q \) is even, we get the same result as in Proposition 2.5 using equivalent method. Also if \( p = 1, 3, 7 \) and \( q \) is even, we obtain the same result as that of Proposition 2.6.

Since \( M_{p, q}^+ \) denotes the subgroup of \( M_{p, q} \) consisting of diffeomorphisms of \( S^p \times S^q \# S^p \times S^q \) which induce identity map on all homology groups, it follows that \( M_{p, q}^+ \) is the kernel of the homomorphism \( \hat{\psi} \). We now compute \( M_{p, q}^+ \). We define a homomorphism

\[
G: \widetilde{\pi}_0(M_{p, q}^+) \longrightarrow \pi_p SO(q+1)
\]

Given an element \( [f] \in \widetilde{\pi}_0(M_{p, q}^+) \), since \( \hat{\psi}(f) \) is identity, it means that if \( i(S^p \times \{p_0\}) \) is the usual identity embedding of \( S^p \times \{p_0\} \) into \( S^p \times S^q \# S^p \times S^q \) where \( p_0 \) is a fixed point in \( S^q \) far away from the connected sum, then the sphere \( S^p \times \{p_0\} \) in \( S^p \times S^q \# S^p \times S^q \) represents a generator of the homology \( H_p(S^p \times S^q \# S^p \times S^q) \approx \mathbb{Z} \otimes \mathbb{Z} \). Since \( \hat{\psi}(f) \) is identity, it follows that \( f(S^p \times p_0) \) is homologous to \( i(S^p \times p_0) \) and since \( p < q \) and by Hurewicz theorem, \( f \) and \( i \) are homotopic and in fact with the dimension restriction, they are diffeotopic. By tubular neighborhood theorem, \( f \) is diffeotopic to a map say \( f'' \) such that \( f''(S^p \times D^q) = S^p \times S^q \) where \( f''(x, y) = (x, \alpha(f''(x)) \cdot y) \) and \( \alpha(f'') : S^p \longrightarrow SO(q) \). Let \( i : SO(q) \longrightarrow SO(q+1) \) be the inclusion map and \( i_* : \pi_p SO(q) \longrightarrow \pi_p SO(q+1) \) the induced map on the homotopy groups. Then we define

\[
G[f] = i_* (\alpha(f''))
\]

**LEMMA 2.7** \( G \) is well-defined.

**PROOF:** Let \( f, h \in M_{p, q}^+ \) such that \( f \) and \( h \) are pseudo-diffeotopic then \( f \cdot h^{-1} \in M_{p, q}^+ \) is pseudo-diffeotopic to the identity. If \( G[f] = i_* \alpha(f'') \) and
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Let \( f(x,y) = (x, \alpha(f')(x) \cdot y) \) and \( h(x,y) = (x, \alpha(h')(x) \cdot y) \) for \((x,y) \in \mathbb{S}^p \times \mathbb{S}^q\) then it follows that

\[
f^{-1}(x,y) = (x, \alpha(f')(x) \cdot y) \quad (x,y) \in \mathbb{S}^p \times \mathbb{S}^q.
\]

We wish to show that \( i_* \alpha(f') = i_* \alpha(h') \). Since \( G(f) = i_* \alpha(f') \in \pi_p \mathbb{S}^{p+1} \) and \( G(h) = i_* \alpha(h') \in \pi_p \mathbb{S}^{p+1} \) then we can define maps \( f_1, h_1 \in \text{Diff}(\mathbb{S}^p \times \mathbb{S}^q) \) such that

\[
f_1(x,y) = (x, \alpha(f')(x) \cdot y) \quad \text{and} \quad h_1(x,y) = (x, \alpha(h')(x) \cdot y)
\]

then consider \( f_1^{-1} h_1 \in \text{Diff}(\mathbb{S}^p \times \mathbb{S}^q) \). Since \( f^{-1} h^{-1} \) is pseudo-diffeotopic to identity so is \( f_1 h^{-1} \) by its definition.

Hence \( f_1^{-1} h_1 \in \text{Diff}(\mathbb{S}^p \times \mathbb{S}^q) \) is diffeotopic to the identity hence it extends to a diffeomorphism \( g \in \text{Diff}(\mathbb{S}^p \times \mathbb{S}^q) \) such that \( g|\text{Diff}(\mathbb{S}^p \times \mathbb{S}^q) = f_1^{-1} h_1 \). Let \( S_\beta \) denote the q-sphere bundle over \( p+1 \)-sphere with characteristic map \( \beta : \mathbb{S}^p \rightarrow \mathbb{S}^{p+1} \). Then we have

\[
S = \mathbb{S}^{p+1} \times \mathbb{S}^q = \mathbb{D}^{p+1} \times \mathbb{S}^q \cup \mathbb{D}^{p+1} \times \mathbb{S}^q
\]

so this gives a q-sphere bundle over a \( p+1 \)-sphere with the characteristic class of the equivalent plane bundle being \( i_* \alpha(f') \cdot i_* \alpha(h')^{-1} \). However, \( f_1^{-1} h_1 \) extends to \( g \in \text{Diff}(\mathbb{S}^p \times \mathbb{S}^q) \) then we have

\[
S = \mathbb{S}^{p+1} \times \mathbb{S}^q = \mathbb{D}^{p+1} \times \mathbb{S}^q \cup \mathbb{D}^{p+1} \times \mathbb{S}^q
\]

Hence we define a map \( H : \mathbb{S}^{p+1} \times \mathbb{S}^q \rightarrow S \)

\[
H(x,y) = \begin{cases} 
(x,y) & \text{if } (x,y) \in \mathbb{D}^{p+1} \times \mathbb{S}^q \\
(x,y) & \text{if } (x,y) \in \mathbb{D}^{p+1} \times \mathbb{S}^q
\end{cases}
\]

\( H \) is well-defined and is a diffeomorphism. This means that \( S = i_* \alpha(f') \cdot i_* \alpha(h')^{-1} \) is a trivial q-sphere bundle over \( S = i_* \alpha(f') \cdot i_* \alpha(h')^{-1} \).

It then follows from \([1, \text{Lemma 3.6(b)}]\) that \( i_* \alpha(f') = i_* \alpha(h') \). Hence \( G \) is well-defined. It is easy to see that \( G \) is a homomorphism.

**Lemma 2.8** \( G(\pi_0(\mathbb{M}^+_{p,q})) = i_* (\pi_p(\mathbb{S}^{p+1})) \).

**Proof:** By the definition of \( G \), \( G(\pi_0(\mathbb{M}^+_{p,q})) = i_* (\pi_p(\mathbb{S}^{p+1})) \) we then show that \( i_* (\pi_p(\mathbb{S}^{p+1})) \subseteq G(\pi_0(\mathbb{M}^+_{p,q})) \). If \( \alpha \in \pi_p(\mathbb{S}^{p+1}) \) then \( a = \alpha \) where \( a : \mathbb{S}^p \rightarrow \mathbb{S}^{p+1} \) then we can define \( f \in \pi_p(\mathbb{M}^+_{p,q}) \) by

\[
f(x,y) = \begin{cases} 
(x, a(x) \cdot y) & \text{if } (x,y) \in (\mathbb{S}^p \times \mathbb{S}^q)_1 \\
(x,y) & \text{if } (x,y) \in (\mathbb{S}^p \times \mathbb{S}^q)_2
\end{cases}
\]

since \( a \in \pi_p(\mathbb{M}^+_{p,q}) \) then \( f \in \pi_0(\mathbb{M}^+_{p,q}) \) and so \( G(f) = \alpha \in \pi_p(\mathbb{S}^{p+1}) \).

In fact since \( p < q \), then \( \pi_p(\mathbb{S}^q) = 0 \) hence it follows from the exact sequence \( \pi_p \mathbb{S}^1 \rightarrow \pi_p \mathbb{S}^q \rightarrow \pi_p(\mathbb{S}^{p+1}) \rightarrow \pi_p \mathbb{S}^q \rightarrow \cdots \) that \( i_* \) is an epimorphism and so it is easily seen that \( G \) is surjective. Hence the proof.
The next lemma is similar to [6, Lemma 3.3].

**Lemma 2.9** Let \( u \in \ker G \), then there exists a representative \( f \in M^+_{p,q} \) of \( u \) such that \( f \) is identity on \( S^p \times D^q \).

**Proof:** If \( p < q-1 \), then \( \pi_{p+1}(S^q) = 0 \) and also \( \pi_p(S^q) = 0 \) and so it follows from the exact sequence

\[
\cdots \to \pi_{p+1}(S^q) \to \pi_p(S \cap D^q) \to \pi_p(S \cap D^q+1) \to \pi_p(S^q) \to \cdots
\]

that \( \iota^*_p \) is an isomorphism hence if \( u = \{ f \} \in \ker G \) then \( G(u) = \iota^*_p(\alpha(f')) = 0 \) implies \( \alpha(f') = 0 \). Since \( f(x,y) = (x, \alpha(f')(x), y) \) for \( (x,y) \in S^p \times D^q \) then it means \( f(x,y) = (x,y) \) hence \( f \) is identity on \( S^p \times D^q \). However, in general let \( g \in M^+_{p,q} \) be defined thus, if \( S^p \times D^q_+ \), \( S^p \times D^q_- \) are subsets of \( (S^p \times S^q)_1 \), away from the connected sum in \( M^+_{p,q} \), we then define

\[
g(x,y) = \begin{cases} (x, \alpha(f')^{-1}(x), y) & \text{for } (x,y) \in S^p \times D^q_+ \text{ and } S^p \times D^q_- \subset (S^p \times S^q)_1 \\ (x,y) & \text{for } (x,y) \in (S^p \times S^q)_2
\end{cases}
\]

since \( \iota^*_p(\alpha(f')) \in \pi_p(S \cap (S^q_1+1)) \) we define \( g' \in M^+_{p,q} \) by

\[
g'(x,y) = \begin{cases} (x, \iota^*_p(\alpha(f'))^{-1}(x), y) & \text{if } (x,y) \in (S^p \times S^q)_1 \\ (x,y) & \text{if } (x,y) \in (S^p \times S^q)_2
\end{cases}
\]

then \( g \) and \( g' \) are diffeotopic and since \( u \in \ker G \), \( G(u) = 0 = \iota^*_p(\alpha(f')) \) then \( g' \) is pseudo-diffeotopic to the identity and so follows that \( g \) is also pseudo-diffeotopic to the identity in \( M^+_{p,q} \). Then the composition \( g \circ f \) is pseudo-diffeotopic to \( f \) and clearly by the definition of \( g \), \( g \circ f \) keeps \( S^p \times D^q_+ \) fixed and represents \( u \) because it is pseudo-diffeotopic to \( f \). Hence the proof.

We now wish to compute \( \ker G \). To do this, we define a homomorphism

\[
N : \text{Ker } G \longrightarrow \text{ker } (\text{Diff}^+(S^p \times S^q))
\]

and show that \( N \) is surjective. Here we adopt the notation \( \text{Diff}^+(S^p \times S^q) \) to mean the set of all diffeomorphisms of \( S^p \times S^q \) to itself which induce identity on all homology groups. Given \( u \in \text{Ker } G \), let \( f \in M^+_{p,q} \) be its representative then it follows from Lemma 2.9 that we can take \( f \) to be identity on \( S^p \times D^q \). So we have a map

\[
f : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \longrightarrow (S^p \times S^q)_3 \# (S^p \times S^q)_4 \,
\]

such that \( f \) is identity on \( S^p \times D^q \subset (S^p \times S^q)_1 \).

Using the technique introduced by Milnor [9] and [3], we perform the spherical modification on the domain \( (S^p \times S^q)_1 \# (S^p \times S^q)_2 \) that removes \( S^p \times D^q \subset (S^p \times S^q)_1 \) and replaces it with \( S^p+1 \times S^q-1 \). Clearly we obtain \( (S^p \times S^q)_2 \) since \( S^p \times D^q \subset S^p+1 \times S^q-1 \) is diffeomorphic to \( S^p+q \). Since \( f \) is the identity on \( S^p \times D^q \), we can assume that \( f(S^p \times D^q) = S^p \times D^q \subset (S^p \times S^q)_3 \) and then perform the corresponding spherical modification on the range \( (S^p \times S^q)_3 \# (S^p \times S^q)_4 \) to obtain \( (S^p \times S^q)_4 \). After this modification we are then left with a diffeomorphism say \( f' \) of \( (S^p \times S^q)_1 \) onto \( (S^p \times S^q)_4 \), i.e., \( f' \in \text{Diff}(S^p \times S^q) \) since \( f \in M^+_{p,q} \) then \( f' \in \text{Diff}^+(S^p \times S^q) \).

So we define \( N[f] = \{ f' \} \).
**LEMMA 2.10** N is well-defined.

**Proof:** Let \( f, g \in \text{Ker } G \) such that \( f \) is pseudo-diffeotopic to \( g \), then \( f \) is identity on \( S^p \times D^q \) and \( g \) is also identity on \( S^p \times D^q \). Since \( f \) is pseudo-diffeotopic to \( g \) then there exists a diffeomorphism

\[
F \in \text{Diff}((S^p \times S^q \# S^p \times S^q) \times I)
\]

such that \( F \) is identity on \( S^p \times D^q \times I \) and \( F | (S^p \times S^q \# S^p \times S^q) \times 0 = f \) while \( F | (S^p \times S^q \# S^p \times S^q) \times 1 = g \). If we now perform the spherical modification on the domain \( (S^p \times S^q)_1 \# (S^p \times S^q)_2 \times I \) of \( F \) by removing \( S^p \times D^q \times I \subset (S^p \times S^q)_1 \times I \) and replacing it with \( S^p \times D^q \times I \subset (S^p \times S^q)_2 \times I \), we then obtain the manifold \( (S^p \times S^q)_2 \times I \) and since \( F \) is identity on \( S^p \times D^q \times I \), we then perform the corresponding modification on the range \( (S^p \times S^q)_2 \# (S^p \times S^q)_4 \times I \) by removing \( S^p \times D^q \times I \subset (S^p \times S^q)_3 \times I \) and replacing it with \( D^p+1 \times S^q \times I \) to obtain \( (S^p \times S^q)_4 \times I \). We then obtain a diffeomorphism

\[
F' : (S^p \times S^q) \times I \longrightarrow (S^p \times S^q) \times I
\]

i.e., \( F' \in \text{Diff}^+((S^p \times S^q) \times I) \) hence \( N(F) = F' \) and \( F' | (S^p \times S^q) \times 0 = f' \) and \( F' | (S^p \times S^q) \times 1 = g' \) hence \( f' \) is pseudo-diffeotopic to \( g' \) and so \( N \) is well-defined.

It is easy to see that \( N \) is a homomorphism.

**LEMMA 2.11** \( N \) is surjective.

**Proof:** Let \( h' \in \text{Diff}^+((S^p \times S^q)) \), we need to find a diffeomorphism \( h \in \mathbb{M}^p_{q,q} \) such that \( N(h) = h' \). If \( D^{p+q} \) is a disc in \( S^p \times S^q \) then we can assume \( h' \) is identity on \( D^{p+q} \) then we have \( h' \in \text{Diff}^+((S^p \times S^q) - D^{p+q}) \). We then define \( h \in \mathbb{M}^p_{q,q} \) thus

\[
h(x, y) = \begin{cases} 
(x, y) & \text{if } (x, y) \in (S^p \times S^q)_1 - D^{p+q} \\
h'(x, y) & \text{if } (x, y) \in (S^p \times S^q)_2 - D^{p+q}
\end{cases}
\]

where \( \mathbb{M}^p_{q,q} = \text{Diff}^+((S^p \times S^q)_1 \# (S^p \times S^q)_2 \times I) \) as earlier stated. \( h \) is well-defined and \( h \in \mathbb{M}^p_{q,q} \). Since \( h \) is identity on \( (S^p \times S^q)_1 \) then it is identity on \( S^p \times D^q \subset (S^p \times S^q)_1 \) hence \( h \in \text{Ker } G \) and clearly \( N(h) = h' \) and so \( N \) is surjective.

We recall from \([6, \S 3]\) the homomorphism

\[
B : \pi_0 \text{Diff}^+((S^p \times S^q)) \longrightarrow \pi_0 \text{SO}(q+1)
\]

which is similarly defined as homomorphism \( G \) and where Sato gave a computation of \( \text{Ker } B \). We will apply this result of \( \text{Ker } B \) to the next lemma.

**LEMMA 2.12** \( \text{Ker } N \) is in one-to-one correspondence with \( \text{Ker } B \).

**Proof:** Let \( f \in \text{Ker } B \), we will produce a diffeomorphism \( f' \in \mathbb{M}^p_{q,q} \) such that \( f' \in \text{Ker } N \). Since \( f \in \text{Ker } B \) then \( f \in \text{Diff}^+((S^p \times S^q)) \) and \( f | S^p \times D^q = \text{identity} \). We define a diffeomorphism \( f' : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \times I \longrightarrow (S^p \times S^q)_3 \# (S^p \times S^q)_4 \times I \) by

\[
f'(x, y) = \begin{cases} 
f(x, y) & \text{if } (x, y) \in (S^p \times S^q)_1 - D^{p+q} \\
(x, y) & \text{if } (x, y) \in (S^p \times S^q)_2 - D^{p+q}
\end{cases}
\]

\( f' \) is well-defined and \( f' \in \mathbb{M}^p_{q,q} \). Since \( f' = f \) on \( (S^p \times S^q)_1 \), and since \( f | (S^p \times D^q) \subset (S^p \times S^q)_1 \) is identity then it follows that \( f' | (S^p \times D^q) = \text{identity} \) and so \( f' \in \text{Ker } G \). However, using \( S^p \times D^q \subset (S^p \times S^q)_1 \) to perform spherical modification on both sides of the domain and range of \( f' \) and the fact that \( f' \) is the identity on \( (S^p \times S^q)_2 \) we clearly see that \( N(f') = \text{identity} \in \text{Diff}((S^p \times S^q)_2 \times I) \) hence \( f' \in \text{Ker } N \).
Conversely let \( f \in \text{Ker } N \), then \( N(f) = f' \in \tilde{\pi}_0^{\text{Diff}^+(S^p \times S^q)} \). We want to show that \( f' \in \text{Ker } B \). Since \( f \in \text{Ker } N \) then it means the image of \( f \) under \( N \) is trivial hence \( N(f) = f' \) is pseudo-diffeotopic to the identity. We now consider \( B(f') \) where \( B : \tilde{\pi}_0^{\text{Diff}^+(S^p \times S^q)} \rightarrow \pi_p^{\text{SO}(q+1)} \) is defined in [6] similar to our homomorphism \( G \). Since \( f' \in \text{Diff}^+(S^p \times S^q) \) and \( p < q \) then \( f'|S^p \times S^q = S^p \times S^q \) where \( f'(x,y) = (x, b(f')(x), y) \) for \( (x,y) \in S^p \times S^q \) and \( b(f') : S^p \rightarrow \text{SO}(q) \). If \( i : \text{SO}(q) \rightarrow \text{SO}(q+1) \) is the inclusion map and \( i_* : \pi_p^{\text{SO}(q)} \rightarrow \pi_p^{\text{SO}(q+1)} \) is the induced homomorphism then \( B(f') = i_* b(f') \in \pi_p^{\text{SO}(q+1)} \).

However since \( f' \) is pseudo-diffeotopic to the identity then let \( H : S^p \times S^q \rightarrow S^p \times S^q \times I \) be the pseudo-diffeotopy between \( f' \) and identity \( \text{id} \).

Then \[
\begin{align*}
D^{p+1} \times S^q & \rightarrow D^{p+1} \times S^q \rightarrow D^{p+1} \times S^q \times I \\
\text{id} & \rightarrow \text{id} \\
\text{id}_1 & \rightarrow \text{id}_2 \\
\end{align*}
\]
is the required diffeomorphism between \( D^{p+1} \times S^q \times I \rightarrow D^{p+1} \times S^q \times S^q \). where \( \text{id}_1(x, y, z) = (x,y,1), \text{id}_2(x, y, z) = (x,y) \) and \( \text{id}_2(x, y, z) = (x,y,1) = (x,y) \). However, consider \( S^{q+1}_b(f') \) the q-sphere bundle over a \((p+1)\)-sphere whose characteristic class of the equivalent normal bundle is \( i_* b(f') \in \pi_p^{\text{SO}(q+1)} \) hence \( S^{q+1}_b(f') = D^{p+1} \times S^q \cup D^{p+1} \times S^q \approx S^{p+1} \times S^q \) by the above diffeomorphism and since \( p < q \) it follows by [1, Prop. 3.6] that \( i_* b(f') = 0 \).

Hence \( f' \in \text{Ker } B \) and so \( \text{Ker } B \) is in one-to-one correspondence with \( \text{Ker } B \).

Since \( N \) is surjective by Lemma 2.11 then we have

**Lemma 2.13** The order of the group \( \text{Ker } G \) equals the order of the direct sum group

\[
\text{Ker } B \oplus \tilde{\pi}_0^{\text{Diff}^+(S^p \times S^q)}
\]

Also since \( G \) is surjective by Lemma 2.8 then it is easily seen that

**Lemma 2.14** The order of \( \tilde{\pi}_0^{\text{Diff}^+(M^+_{p,q})} \) is equal to the order of the direct sum group

\[
\pi_p^{\text{SO}(q+1)} \oplus \text{Ker } B \oplus \tilde{\pi}_0^{\text{Diff}^+(S^p \times S^q)}
\]

However one can easily deduce from [6, §4]

**Lemma 2.15** \( \text{ker } B \approx \pi_q^{\text{SO}(p+1)} \oplus \emptyset^{p+q+1} \)

Also from [6, Thm. II] and [1, Thm. 3.10] we have

**Lemma 2.16** \( \tilde{\pi}_0^{\text{Diff}^+(S^p \times S^q)} = \pi_p^{\text{SO}(q+1)} \oplus \pi_q^{\text{SO}(p+1)} \oplus \emptyset^{p+q+1} \)

Combining Lemmas 2.12, 2.13, 2.14, 2.15, and 2.16, we obtain

**Theorem 2.17** For \( p < q \), the order of the group \( \tilde{\pi}_0^{\text{Diff}^+(M^+_{p,q})} \) equals twice the order of the group \( \pi_p^{\text{SO}(q+1)} \oplus \pi_q^{\text{SO}(p+1)} \oplus \emptyset^{p+q+1} \).

3. CLASSIFICATION OF MANIFOLDS

Consider the class of manifolds \( \{M, \lambda_1, \lambda_2\} \) where \( M \) is a manifold of type
(n, p, 2) where \( n = p+q+1 \) and \( p = 3, 5, 6, 7 \) (mod 8) and \( \lambda_1, \lambda_2 \) are the generators of \( H_p(M) = \mathbb{Z} \oplus \mathbb{Z} \). By the proof of Theorem 1.1 we have an embedding \( \varphi_i : S^p \times D^{q+1} \to M \) which represents the homology class \( \lambda_i \) \( i = 1, 2 \). If we then take the connected sum along the boundary of the two embedded copies of \( S^p \times D^{q+1} \) we have an embedding

\[
i : S^p \times D^{q+1} \# S^p \times D^{q+1} \to M
\]

such that \( i_*[S^p] = \lambda_1 + \lambda_2 \)

Two of such manifolds \( \{M, \lambda_1, \lambda_2\} \) and \( \{M', \lambda_1', \lambda_2'\} \) will be said to be equivalent if there is an orientation preserving diffeomorphism of \( M \) onto \( M' \) which takes \( \lambda_1 \) to \( \lambda_1' \) \( i = 1, 2 \). Let \( \mathcal{M}_n \) be the equivalent class of manifolds satisfying these conditions. This equivalent class which is also the diffeomorphism class has a group structure. The operation is connected sum along the boundary \( S^p \times S^{q+1} \# S^p \times S^q \) of \( S^p \times D^{q+1} \# S^p \times D^{q+1} \). If \( \{M, \lambda_1, \lambda_2\}, \{M', \lambda_1', \lambda_2'\} \in \mathcal{M}_n \), then let

\[
i_1 : S^p \times D^{q+1} \# S^p \times D^{q+1} \to M
\]

be an orientation preserving embedding such that

\[
i_1_*[S^p] = \lambda_1 + \lambda_2 \]

and since there is an orientation reversing diffeomorphism of \( S^p \times D^{q+1} \# S^p \times D^{q+1} \) to itself (because \( S^p \times D^{q+1} \) is a trivial \( q+1 \)-disc bundle over \( S^p \)) then we have an orientation reversing embedding \( i_2 : S^p \times D^{q+1} \# S^p \times D^{q+1} \to M' \) such that

\[
i_2_*[S^p] = \lambda_1' + \lambda_2'.
\]

We now obtain \( M \# M' \) from the disjoint sum

\[
(M - \text{Int } i_1(S^p \times D^{q+1} \# S^p \times D^{q+1})) \cup (M' - \text{Int } i_2(S^p \times D^{q+1} \# S^p \times D^{q+1}))
\]

by identifying \( i_1(x) \) with \( i_2(x) \) for \( x \in S^p \times S^q \# S^p \times S^q \). We will call this operation the connected sum along double p-cycle. Where the \( 2p \) in \( M \# M' \) means that we are identifying along the boundary of embedded copies of connected sum along the boundary of two copies of \( S^p \times D^{q+1} \). It is easy to see that \( H_p(M \# M') \approx \mathbb{Z} \oplus \mathbb{Z} \). Since we have identified \( i_1_*[S^p] \) with \( i_2_*[S^p] \) we can define

\[
i_*[S^p] = \lambda_1 + \lambda_2 \]

the generators of \( H_p(M \# M') \) then we see that \( M \# M' \in \mathcal{M}_n \).

**Lemma 3.1** The connected sum along the double p-cycle is well-defined and associative.

**Proof:** We need to show that the operation does not depend on the choice of the embeddings. Suppose there is another embedding \( \varphi'_i : S^p \times D^{q+1} \to M \) which represents the homology class \( \lambda_i \) \( i = 1, 2 \) and gives a corresponding embedding \( i'_1 : S^p \times D^{q+1} \# S^p \times D^{q+1} \to M \). By the tubular neighborhood theorem \( \varphi'_1(S^p \times D^{q+1}) \) and \( \varphi'_1(S^p \times D^{q+1}) \) differ only by rotation of their fiber, i.e., by an element of \( \pi_p SO(q+1) = 0 \) since \( p = 3, 5, 6, 7 \) (mod 8) hence the two embeddings are isotopic and so the corresponding embeddings

\[
i_1 : S^p \times D^{q+1} \# S^p \times D^{q+1} \to M \quad \text{and} \quad i'_1 : S^p \times D^{q+1} \# S^p \times D^{q+1} \to M
\]

are isotopic.

The definition does not therefore depend on the choice of \( i_1 \). With similar argument it does not depend on \( i_2 \). The connected sum is therefore well-defined.

Associativity is easy to check.
LEMMA 3.2 If \([M, \lambda_1, \lambda_2], [M_1, \lambda_1, \lambda_2] \in \mathcal{M}_n\), such that they are equivalent. If \([M', \lambda_1', \lambda_2'] \in \mathcal{M}_n\) then \((M, #M', \lambda_1, \lambda_2, \# \lambda_1', \# \lambda_2')\) is equivalent to \((M_2, #M', \lambda_1', \lambda_2, \# \lambda_1', \# \lambda_2')\).

PROOF: Since \(M, M_1\) are equivalent in \(\mathcal{M}_n\) then there exists an orientation preserving diffeomorphism \(f: M \rightarrow M_1\) which carries \(\lambda_1\) to \(\lambda_1\) and \(\lambda_2\) to \(\lambda_2\) hence it carries the embedding \(\varphi_i(S^p \times D^{q+1})\) to the corresponding embedding \(\varphi_i'(S^p \times D^{q+1})\) \(i = 1, 2\) and so \(f\) carries the embedding \(i(S^p \times D^{q+1} # S^p \times D^{q+1}) \subset M\) to the embedding \(i'(S^p \times D^{q+1} # S^p \times D^{q+1}) \subset M_1\) hence \(f\) induces a diffeomorphism

\[
f': M - Int i(S^p \times D^{q+1} # S^p \times D^{q+1}) \rightarrow M_1 - Int i'(S^p \times D^{q+1} # S^p \times D^{q+1})
\]

which carries \(\lambda_1\) to \(\lambda_1\) and \(\lambda_2\) to \(\lambda_2\).

Trivially we have the identity map

\[
id: M' - Int i'(S^p \times D^{q+1} # S^p \times D^{q+1}) \rightarrow M' - Int i(S^p \times D^{q+1} # S^p \times D^{q+1})
\]

which carries \(\lambda_1\) to \(\lambda_1\) and \(\lambda_2\) to \(\lambda_2\). We then take the connected sum along their boundary \(S^p \times S^q \# S^p \times S^q\) to have \(M \# M'\) which is disjoint sum of \(2p\)

\[M - Int i(S^p \times D^{q+1} # S^p \times D^{q+1}) \cup M' - Int i'(S^p \times D^{q+1} # S^p \times D^{q+1})\]

by identifying \(i(x)\) and \(i'(x)\) for \(x \in S^p \times S^q \# S^p \times S^q\). Similarly \(M_1 \# M_2\) is the disjoint sum of

\[M_1 - Int i_1(S^p \times D^{q+1} # S^p \times D^{q+1}) \cup M' - Int i_1'(S^p \times D^{q+1} # S^p \times D^{q+1})\]

by identifying \(i_1(x)\) and \(i_1'(x)\) for \(x \in S^p \times S^q \# S^p \times S^q\). Clearly we have a diffeomorphism

\[g: M \# M' \rightarrow M_1 \# M_2\]

which is \(f'\) on \(M\) and identity of \(M'\) and \(g\) carries \(\lambda_1, \# \lambda_1\) to \(\lambda_1, \# \lambda_1\) and \(\lambda_2, \# \lambda_2\) to \(\lambda_2, \# \lambda_2\). Hence \([M \# M', \lambda_1, \# \lambda_1, \lambda_2, \# \lambda_2]\) is equivalent to \([M_1 \# M_2, \lambda_1, \# \lambda_1, \lambda_2, \# \lambda_2]\) in \(\mathcal{M}_n\). That proves the lemma.

If we now take two copies of \(S^p \times D^{q+1} # S^p \times D^{q+1}\) and identify the two copies on their common boundaries by the identity map, we will obtain the manifold \(S^p \times S^q \# S^p \times S^q\), i.e., \(S^p \times S^q \# S^p \times S^q = (S^p \times D^{q+1} # S^p \times D^{q+1}) \cup (S^p \times D^{q+1} # S^p \times D^{q+1})\) where \(id=\text{identity: } S^p \times S^q \# S^p \times S^q \rightarrow S^p \times S^q \# S^p \times S^q\). If \(\lambda_0, \lambda_0\) are the generators of \(H_p(S^p \times S^q # S^p \times S^q) = \mathbb{Z} \oplus \mathbb{Z}\) and \(-\lambda_1 + (-\lambda_2) \in H_p(-M) = \mathbb{Z} \oplus \mathbb{Z}\) where \(i_1[S^p] = -\lambda_1 + \lambda_2\) and \(i: M \rightarrow -M\) is the orientation reversing diffeomorphism then we have the following.

LEMMA 3.3 \(\mathcal{M}_n\) is a group with identity element \((S^p \times S^{q+1} # S^p \times S^{q+1}, \lambda_0, \lambda_0)\) and for \((M, \lambda_1, \lambda_2) \in \mathcal{M}_n\) \((-M, -\lambda_1, -\lambda_2)\) is the inverse element.

To be able to prove our main theorem later, we need to investigate \(\tilde{\text{Diff}}(S^p \times D^{q+1} # S^p \times D^{q+1})\). As in the case of \(\tilde{\text{Diff}}(M, p, q)\), we define a homomorphism \(\tilde{\phi} : \tilde{\text{Diff}}(S^p \times D^{q+1} # S^p \times D^{q+1}) \rightarrow \text{Auto}\,(H_*(S^p \times D^{q+1} # S^p \times D^{q+1}))\) by induced automorphism of homology groups. Since \(S^p \times D^{q+1} # S^p \times D^{q+1}\) has the homotopy type of \(S^p \times D^{q+1} \vee S^p \times D^{q+1}\) then

\[
H_\ast(S^p \times D^{q+1} # S^p \times D^{q+1}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = p \end{cases}
\]
Using similar ideas in §2, it is easy to prove the following.

**Lemma 3.4**

\[
\tilde{\tau}'(\tilde{\alpha}_0(Diff(S^p \times D^{q+1} # S^p \times D^{q+1}))) = \begin{cases} 
    \mathbb{Z}_4 & \text{if } p \text{ is even} \\
    GL(2, \mathbb{Z}) & \text{if } p = 1, 3, 7 \\
    \mathbb{H} & \text{if } p \text{ is odd but } \neq 1, 3, 7
\end{cases}
\]

Let \( \tilde{\alpha}_0(Diff(S^p \times D^{q+1} # S^p \times D^{q+1})) \) be the set of all diffeomorphisms of \( S^p \times D^{q+1} # S^p \times D^{q+1} \) which induce identity automorphisms on its homology. Then it follows that \( \tilde{\tau}'_0(Diff(S^p \times D^{q+1} # S^p \times D^{q+1})) \) is the kernel of \( \tilde{\tau}' \). We define a homomorphism

\[
G' : \pi_0 SO(q+1) \longrightarrow \tilde{\tau}_0 Diff\_r^+(S^p \times D^{q+1} # S^p \times D^{q+1})
\]

If \( \alpha \in \pi_0 SO(q+1) \) and \( \alpha = [a] \) then we define a map

\[
g_a : S^p \times D^{q+1} # S^p \times D^{q+1} \longrightarrow S^p \times D^{q+1} # S^p \times D^{q+1}
\]

by

\[
g_a(x, y) = \begin{cases} 
    (x, a(x), y) & \text{for } (x, y) \in (S^p \times D^{q+1})_1 \\
    (x, a(x), y) & \text{for } (x, y) \in (S^p \times D^{q+1})_2
\end{cases}
\]

\( g_a \) is clearly well-defined and it is a diffeomorphism and since \( g_a \) keeps \( S^p \) fixed, it induces identity on all homology groups hence \( g_a \in Diff\_r^+(S^p \times D^{q+1} # S^p \times D^{q+1}) \).

We will define \( G'([\alpha]) = \{g_a\} \)

**Lemma 3.5** \( G' \) is well defined.

**Proof:** If \( a' \in \pi_0 SO(q+1) \) such that \( a \) is homotopic to \( a' \) and let \( H : S^p \times I \longrightarrow SO(q+1) \) be the homotopy such that \( H(S^p \times 0) = a \) and \( H(S^p \times 1) = a' \) then we construct a diffeomorphism \( F \) of \( (S^p \times D^{q+1} # S^p \times D^{q+1}) \times I \) by

\[
F(x, y, t) = \begin{cases} 
    (x, H(x, t), y) & (x, y) \in (S^p \times D^{q+1})_1 \\
    (x, H(x, t), y) & (x, y) \in (S^p \times D^{q+1})_2
\end{cases}
\]

This is the diffeotopy which connects \( g_a \) and \( g_{a'} \).

**Lemma 3.6** \( G' \) is surjective.

**Proof:** Let \( \{f\} \in \tilde{\tau}_0 Diff\_r^+(S^p \times D^{q+1} # S^p \times D^{q+1}) \) then \( f \) induces identity on all homology groups. However \( H_\delta(S^p \times D^{q+1} # S^p \times D^{q+1}) \approx \mathbb{Z} \oplus \mathbb{Z} \) and so if \( \lambda_1 \) and \( \lambda_2 \) represents the generators of the first and second summand and the embeddings

\[
i_1 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} # S^p \times D^{q+1} \quad \text{and} \quad i_2 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} # S^p \times D^{q+1}
\]

represents the homology class \( \lambda_1 \) and \( \lambda_2 \) respectively, since \( f \) induces identity on homology then \( f(S^p \times \{p_0\}) \) and \( i_1(S^p \times \{p_0\}) \) are homologous. Since \( p < q \) and by Hurewicz theorem \( i_1 \) and \( f \circ i_1 \) are homotopic, by Haefliger [10] and by the diffeotopy extension theorem and tubular neighborhood theorem, there exists \( f' \) in the diffeotopy class of \( f \) such that \( f'(x, y) = (x, a(x), y) \) for \( (x, y) \in (S^p \times D^{q+1})_1 \) where \( S^p \times D^{q+1} \) is the tubular neighborhood of \( S^p \times \{p_0\} \) and \( a : S^p \longrightarrow SO(q+1) \). Similar argument applies to the embedding \( i_2 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} # S^p \times D^{q+1} \) and
so we have a map \( f" \) in the diffeotopy class of \( f \) hence in the diffeotopy class of \( f' \) and so \( f" \) must be of the form \( f"(x,y) = (x,a(x) \cdot y) \) where \( (x,y) \in (S^p \times D^{q+1})_2 \).

It follows that

\[
((x,a(x) \cdot y) \quad (x,y) \in (S^p \times D^{q+1})_1
\]

\[
((x,a(x) \cdot y) \quad (x,y) \in (S^p \times D^{q+1})_2
\]

Hence \( G' \) is surjective.

One can easily deduce from Lemma 3.6 that \( \tilde{\pi}_0 \text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}) \) is a factor group of \( \pi_p(SO(p+q+1)) \).

**THEOREM 3.7** Let \( M \) be an \( n \)-dimensional closed simply connected manifold of type \((n,p,2)\) where \( n = p+q+1 \) with \( p = 3,5,6,7 \) \((\mod 8)\) then the number of differentiable manifolds satisfying the above conditions up to diffeomorphism is twice the order of the direct sum group \( \pi_p(SO(p+q+1)) \).

**PROOF:** We define a map \( C : \tilde{\pi}_0(M^+_{p,q}) \rightarrow \mathcal{M}_n \) and show that \( C \) is an isomorphism. Let \( \{f\} \in \pi_0(M^+_{p,q}) \) then \( f \) is a diffeomorphism of \( S^p \times S^q \# S^p \times S^q \) which induce identity on homology. We then take two copies \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \) and \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \) of \( S^p \times D^{q+1} \# S^p \times D^{q+1} \) and attach them on the boundary by \( f \) to have \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \). An orientation is chosen to be compatible with \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \) and the manifold obtained belongs to the group \( \mathcal{M}_n \). The generators of the \( p \)-dimensional homology group is fixed to be the one represented by the usual embedding \( S^p \times \{p_0\} \rightarrow (S^p \times D^{q+1})_1 \subset (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \) and \( S^p \times \{p_0\} \rightarrow (S^p \times D^{q+1})_2 \subset (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \). We then define

\[
C[f] = (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \cdot \mathcal{M}_n \text{ where } \mathcal{M}_n \text{ is well-defined.}
\]

Let \( f_0, f_1 \in M^+_{p,q} \) such that \( f_0 \) is pseudo-diffeotopic to \( f_1 \) then there exists a homotopy \( H : (S^p \times S^q \# S^p \times S^q) \times I \rightarrow (S^p \times S^q \# S^p \times S^q) \times I \) such that \( H(x,y,0) = f_0 \) and \( H(x,y,1) = f_1 \) then we wish to show that \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \) is diffeomorphic to \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \). We then define a map

\[
(f_0, f_1, H, \text{id}_0, \text{id}_1) \rightarrow \mathcal{M}_n
\]

where \( \text{id}_0(x,y) = (x,y,1), \text{id}_1(x,y,0) = (x,y), f_0(x,y,0) = f_0(x,y) \) and \( f_1(x,y) = f_1(x,y,1) \).

This is a well-defined map and is the required diffeomorphism from \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \) to \( (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \). Hence \( C \) is well-defined and it is easy to see that \( C \) is a homomorphism. By Theorem 1.1 it follows that \( C \) is surjective. We now need to show that \( C \) is injective. Suppose \( \{f\} \in \tilde{\pi}_0(M^+_{p,q}) \) and \( C(f) = (M, \lambda_1, \lambda_2) \) is trivial, then it follows that
M = \( (S^p \times D^{q+1} \# S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \# S^p \times D^{q+1}) \) is diffeomorphic to
\[ (S^p \times D^{q+1} \# S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \# S^p \times D^{q+1}) = S^p \times S^{q+1} \# S^p \times S^{q+1} \] with p-dimensional homology generators \( \lambda_0, \lambda_2 \), by a diffeomorphism \( d \) which carries \( \lambda_1 \) to \( \lambda_0 \) and \( \lambda_2 \) to \( \lambda_2 \), i.e.,
\[
(S^p \times D^{q+1} \# S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})
\]
It is easy to see that since \( d \) carries \( \lambda_1 \) to \( \lambda_0 \) and \( \lambda_2 \) to \( \lambda_2 \) and because \( p = 3, 5, 6, 7 \) (mod 8) then \( d \) is the identity on \( (S^p \times D^{q+1} \# S^p \times D^{q+1}) \). On the boundary \( S^p \times S^{q+1} \# S^p \times S^{q+1} \), \( d \) is just \( f \). Since \( d \) is a diffeomorphism it follows that \( f \) extends to a diffeomorphism of \( (S^p \times D^{q+1} \# S^p \times D^{q+1}) \), which means
\[
f \in \text{Diff}^+(S^p \times S^{q+1} \# S^p \times S^{q+1})
\]
is extendable to \( \text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}) \), but by Lemma 3.5, \( \pi_0 \text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}) \) is a factor group of \( \pi_p (SO(q+1)) \) but since \( p = 3, 5, 6, 7 \), mod 8 then \( \pi_p (SO(q+1)) = 0 \). Hence \( f \) is pseudo-diffeotopic to the identity and so \( C \) is injective. It then follows that \( C \) is an isomorphism. By Theorem 2.17 and since \( p = 3, 5, 6, 7 \) (mod 8) it follows that the order of the group \( \pi_0 (M^{p+1}_{p,q}) \) is twice the order of the group \( \pi_0 SO(p+1) \) and since \( C \) is an isomorphism the theorem is proved. The methods used here if carefully applied can be used to obtain a general result.

**Theorem 3.8** If \( M \) is a smooth, closed simply connected manifold of type \( (n, p, r) \) where \( n = p + q + 1 \) and \( p = 3, 5, 6, 7 \) (mod 8) then the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to \( r \) times the order of \( \pi_0 SO(p+1) \).

**References**