ON BAZILEVIC FUNCTIONS

KHALIDA I. NOOR and SUMAYYA A. AL-BANY
Mathematics Department, Girls College of Education
(Science Section) Sitteen Road, Malaz, Riyadh,
Saudi Arabia.

(Received January 23, 1985 and in revised form June 28, 1986)

ABSTRACT. Let $B(\beta)$ be the class of Bazilevic functions of type $B(\beta>0)$. A function $f \in B(\beta)$ if it is analytic in the unit disc $E$ and $\Re \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)}>0$, where $g$ is a starlike function. We generalize the class $B(\beta)$ by taking $g$ to be a function of radius rotation at most $k(k>2)$. Archlength, difference of coefficient, Hankel determinant and some other problems are solved for this generalized class. For $k=2$, we obtain some of these results for the class $B(\beta)$ of Bazilevic functions of type $\beta$.

KEY WORDS AND PHRASES. Bazilevic functions, functions of bounded boundary rotation; Hankel determinant, close-to-convex functions, radius of $\alpha$-convexity. 1980 MATHEMATICS SUBJECT CLASSIFICATION CODES 30A 32, 30A34.

1. INTRODUCTION.

Bazilevic [1] introduced a class of analytic function $f$ defined by the following relation. For $z \in E$, let $B(\beta)$

$$f(z) = B \frac{\beta}{1+a^2} \left[ \sum_{0}^{\frac{\beta}{1-a^2}} \frac{\beta}{g^{1+a^2}((\xi))d\xi} \right]$$

where $a$ is real, $\beta > 0$, $\Re h(z)>0$ and $g$ belongs to the class $S^*$ of starlike functions. Such functions, he showed, are univalent [1]. With $a=0$ in (1.1), we have for $z \in E$

$$\Re \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)}>0 \quad (1.2)$$

This class of Bazilevic functions of type $\beta$ was considered in [2]. We denote this class of functions by $B(\beta)$. We notice that if $\beta =1$ in (1.2), we have the class $K$ of close-to-convex functions. We need the following definitions.

Definition 1.1

A function $f$ analytic in $E$ belongs to the class $V_k$ of functions with bounded boundary rotation, if $f(0) = 0, f'(0) = 1, f'(z) \neq 0$, such that for $z = re^{i\theta} \in E, 0 < r < 1$
For $k=2$, we obtain the class $C$ of convex functions. It is known [3] that for $2<k<4$, $V_k$ consists entirely of univalent functions. The class $V_k$ has been studied by many authors, see [3], [4], [5] etc.

Definition 1.2

Let $f$ be analytic in $E$ and $f(0)=0, f'(0)=1$. Then $f$ is said to belong to the class $R_k$ of functions with bounded radius rotation, if $z=re^{i\theta} \in E$, $0<r<1$

$$\int_0^{2\pi} |\text{Re} \left( \frac{zf'(z)}{f(z)} \right)| d\theta < k\pi, \quad k \geq 2$$

It is clear that $f \in V_k$ if and only if $zf' \in R_k$. We also note that $R_2 = S^*$.

We now give the following generalized form of the class $B(\beta)$.

Definition 1.3

Let $f$ be analytic in $E$ and $f(0)=1, f'(0)=1$. Then $f$ belongs to the class $B_k(\beta)$, $\beta>0$ if there exists a $g \in R_k, k \geq 2$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1-\beta}{1-\beta} (S_1(z)/z) \left( \frac{k}{4} + \frac{1}{2} \right), \quad k \geq 2$$

We notice that, when $\beta=1$, $B_k(1) \equiv T_k$, a class of analytic functions introduced and discussed in [6]. Also $B_2(\beta) = B(\beta)$ and $B_2(1) = K$, the class of close-to-convex functions.

2. PRELIMINARIES

We shall give here the results needed to prove our main theorems in the preceding section.

Lemma 2.1 [3].

Let $f \in V_k$. Then there exist two starlike functions $S_1, S_2$ such that for $z \in E$

$$f'(z) = \frac{(S_1(z)/z)^{k/4} + \frac{1}{2}}{(S_2(z)/z)^{k/4} - \frac{1}{2}}, \quad k \geq 2$$

Lemma 2.2

Let $H$ be analytic in $E$, $|H(0)| \leq 1$ and be defined as

$$H(z) = (\frac{k}{2} + \frac{1}{2})h_1(z) - (\frac{k}{4} - \frac{1}{2})h_2(z), \quad \text{Re} h_i(z) > 0, \quad i=1,2,k \geq 2$$

Then, for $z = re^{i\theta}$,

$$\int_0^{2\pi} |H(z)|^2 \text{d}\theta < \frac{1-(k^2-1)r^2}{1-r^2}$$

and
This result is known [6] and, for \( k=2 \), we obtain Pommerenke's result [7] for functions of positive real parts.

**Lemma 2.3**

Let \( S_1 \) be univalent in \( E \). Then:

(i) there exists a \( z_1 \) with \( |z_1|=r \) such that for all \( z \), \( |z|=r \)

\[
|z-z_1| |S_1(z)| < \frac{2r^2}{1-r^2}, \quad \text{see} \ [8] \tag{2.4}
\]

and

(ii) \[
\frac{r}{(1+r)^2} < |S_1(z)| < \frac{r}{(1-r)^2}, \quad \text{see} \ [9] \tag{2.5}
\]

**Definition 2.1.**

Let \( f \) be analytic in \( E \) and be given by \( f(z) = z + \sum_{n=1}^{\infty} a_n z^n \). Then the \( q \)th Hankel determinant of \( f \) is defined for \( q>1, \ n>1 \)

\[
H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
a_{n+q-1} & \cdots & \cdots & a_{n+2q-2}
\end{vmatrix} \tag{2.6}
\]

**Definition 2.2.**

Let \( z_1 \) be a non-zero complex number. Then, with \( \Delta_0(n, z_1, f) = a_n \);

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ we define for } j \geq 1, \]

\[
\Delta_j(n, z_1, f) = \Delta_{j-1}(n, z_1, f) - \Delta_{j-1}(n+1, z_1, f) \tag{2.7}
\]

**Lemma 2.4**

Let \( f \) be analytic in \( E \) and let the Hankel determinant of \( f \) be defined by (2.6). Then, writing \( \Delta_j = \Delta_j(n, z_1, f) \), we have

\[
H_q(n) = \begin{vmatrix}
\Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \Delta_q(n+q-1) \\
\Delta_{2q+3}(n+1) & \Delta_{2q-4}(n+2) & \Delta_q(n+q) \\
\vdots & \ddots & \ddots \\
\Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) & \Delta_q(1+n+2q-3)
\end{vmatrix} \tag{2.8}
\]

**Lemma 2.5**

With \( z_1 = \frac{n}{n+1} y \), and \( \nu > 0 \) any integer,

\[
\Delta_j(n+\nu, z_1, zf') = \sum_{k=0}^{j} \binom{j}{k} \frac{y^{k-(k-1)n}}{(n+1)^k} \Delta_{j-k}(n+\nu+k, y, f) \tag{2.9}
\]

Lemmas 2.4 and 2.5 are due to Noonan and Thomas [10].
Lemma 2.6 [11].

Let \( N \) and \( D \) be analytic in \( E \), \( N(0) = D(0) \) and \( D \) maps \( E \) onto many sheeted region which is starlike with respect to the origin. Then \( \Re \frac{N'(z)}{D'(z)} > 0 \) implies \( \Re \frac{N(z)}{D(z)} > 0 \).

3. MAIN RESULTS.

THEOREM 3.1: Let \( f \in B_k(\beta); k \geq 2, \ 0 < \beta < 1 \). Then

\[
\lambda_r(f) \leq C(k, \beta) M^{1-\beta}(r) \left( \frac{1}{1-r} \right) \beta \left( \frac{k}{2} + 1 \right),
\]

where \( C(k, \beta) \) is a constant depending on \( k, \beta \) only. \( \lambda_r(f) \) denotes the length of the closed curve \( f(|z| = r < 1) \) and \( M(r) = \max_{|z| = r} |f(z)| \).

PROOF: We have

\[
\lambda_r(f) = \int_0^{2\pi} |zf'(z)| \, d\theta, \quad z = re^{i\theta}
\]

\[
= \int_0^{2\pi} \left| f^{1-\beta}(z) g^\beta(z) h(z) \right| \, d\theta, \quad \text{using (1.5), where } g \in R_k \text{ and } \Re h(z) > 0.
\]

\[
\leq M^{1-\beta}(r) \int_0^{2\pi} |g^\beta(z) h(z)| \, d\theta
\]

\[
\leq M^{1-\beta}(r) \left\{ \int_0^{2\pi} \int_0^{r} \left| g^\beta(z) h(z) + g^\beta(z) h'(z) \right| \, d\theta \right\}
\]

\[
\leq M^{1-\beta}(r) \left\{ \int_0^{2\pi} \int_0^{r} \left| g^\beta(z) h(z) \right| \, d\theta + \int_0^{2\pi} \int_0^{r} \left| g^\beta(z) h'(z) \right| \, d\theta \right\}
\]

where \( g^\beta(z) = H(z) \) is defined as in Lemma 2.2.

Using Lemma 2.1, Lemma 2.3 (ii), Schwarz inequality and then Lemma 2.2 for both general and special cases \((k \geq 2, k = 2)\), we have

\[
\lambda_r(f) \leq C(k, \beta) M^{1-\beta}(r) \left( \frac{1}{1-r} \right) \beta \left( \frac{k}{2} + 1 \right), \ 0 < \beta \leq 1, \ C(k, \beta) \text{ is a constant depending on } k, \beta \text{ only.}
\]

Corollary 3.1

For \( k = 2, f \in B(\beta) \) and \( \lambda_r(f) \leq C(\beta) M^{1-\beta}(r) \left( \frac{1}{1-r} \right)^2 \beta \)

THEOREM 3.2.

Let \( f \in B_k(\beta), \ 0 < \beta < 1, \ k \geq 2 \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). Then for \( n > 2 \)

\[
|a_n| \leq C_1(\beta,k) M^{1-\beta}(1-\frac{1}{n}) \beta^{\left( \frac{k}{2} + 1 \right)-1},
\]

\( C_1(\beta,k) \) is a constant depending only upon \( k \) and \( \beta \).
PROOF: Since, with \( z = re^{i\theta} \), Cauchy’s theorem gives

\[
\frac{n}{2\pi r^2} \int_0^{2\pi} zf'(z)e^{-in\theta} d\theta,
\]

then

\[
|a_n| \leq \frac{1}{2\pi r^2} \int_0^{2\pi} |zf'(z)| d\theta = \frac{1}{2\pi r^n} L_r(f).
\]

Using theorem 3.1 and putting \( r = 1 - \frac{1}{n} \), we obtain the required result.

Corollary 3.2

When \( \beta = 1, f \in T_k \), and from theorem 3.2 we have

\[
|a_n| \leq A(k)n^{k/2}, \quad A(k) \text{ being a constant depending on } k \text{ only.}
\]

This result was proved in [6].

Corollary 3.3

For \( k = 2, f \in B(\beta) \) and

\[
\left| a_n \right| \leq A_1(\beta) M^{1-\beta} (1-\frac{1}{n}), n \geq 2.
\]

THEOREM 3.3.

Let \( f \) be as defined in theorem 3.2. Then, for \( n \geq 2, k > \frac{5}{\beta} - 2 \),

\[
\left| a_{n+1} - a_n \right| = O(1) M^{1-\beta} (1-\frac{1}{n}) \beta(k+1) + 1 - 2
\]

where \( O(1) \) depends only on \( k \) and \( \beta \).

PROOF: For \( z_1 \in E, n \geq 2 \) and \( z = re^{i\theta} \in E \), we have

\[
\left| (n+1)z_1 |a_{n+1} - a_n| \right| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-z_1| |zf'(z)| d\theta
\]

\[
= \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-z_1| |f(z)|^{1-\beta} \beta \left( \frac{k+2}{4} - 1 \right) \left( \frac{k+2}{4} + 1 \right) \left| h(z) \right| d\theta,
\]

where we have used (1.5).

Taking \( M(r) = \max f(z) \), and using (2.1) (2.4) and (2.5), we have

\[
\left| (n+1)z_1 |a_{n+1} - a_n| \right| \leq \frac{1-\beta(r)}{2\pi r^{n+1}} \left( \frac{k+2}{4} - 1 \right) \left( \frac{k+2}{4} + 1 \right) \left| S_1(z) \right| \left| h(z) \right| d\theta,
\]

where \( S_1 \) is a starlike function.

Schwarz inequality, together with Lemma 2.2 \( (k=2) \) and subordination for starlike functions [12] yields

\[
\left| (n+1)z_1 |a_{n+1} - a_n| \right| \leq \frac{1-\beta(r)}{2\pi r^{n+1}} \left( \frac{k+2}{4} - 1 \right) \left( \frac{k+2}{4} + 1 \right) \left| S_1(z) \right| \left| h(z) \right| d\theta
\]

\[
\leq C(k,\beta) M^{1-\beta}(r) \left( \frac{k+2}{4} + 1 \right) - 1
\]

where \( C(k,\beta) \) is a constant depending only on \( k \) and \( \beta \). Choosing \( |z_1| = r = \frac{n}{n+1} \), we obtain the required result.
Corollary 3.4  
If $\beta=1$, $f \in T_k$ and we obtain a known [6] result, for $k>3$, 
\[ \frac{k}{2} - 1 \]
\[ \|a_{n+1}\| - |a_n| = O(1/n). \]

We now proceed to study the Hankel determinant problem for the class $B_k(\beta)$.

**THEOREM 3.4.**

Let $f \in B_k(\beta)$, $0<\beta<1$, $k \geq 2$ and let the Hankel determinant $H_q(n)$ of $f$ be defined as in definition 2. Then

\[ H_q(n) = 0(1)M^{1-\beta}(r) \left\{ \begin{array}{ll}
\beta(k+1)/2 - 1, & q=1 \\
\beta(k+1)/2 q - q^2, & q>2, k \geq 8q - 8/\beta - 2
\end{array} \right. \]

**PROOF:** Since $f \in B_k(\beta)$, we can write

\[ zf'(z) = f^{1-\beta}(z) g^{\beta}(z) h(z), \ Re h(z)>0, g \in R_k. \]

Let $F(z) = zf'(z)$. Then for $j \geq 1$, $z_1$ any non-zero complex number and $z = re^{i\theta}$, consider $\Delta_j(n, z_1, F)$ as defined by (2.7). Then

\[ |\Delta_j(n, z_1, F)| \leq \frac{1}{2\pi r^n+j} \left\{ \begin{array}{ll}
2\pi \int_0^{2\pi} |F(z)| e^{-i(n+j)\theta} d\theta & q=1 \\
2\pi \int_0^{2\pi} |f^{1-\beta}(z) g^\beta(z) h(z)| d\theta & q \geq 2, k \geq 8q - 8/\beta - 2
\end{array} \right. \]

Using (2.1), (2.4) and (2.5), we have

\[ |\Delta_j(n, z_1, F)| \leq C(k, \beta, J)M^{1-\beta}(r) \left( \begin{array}{ll}
\frac{k-2}{4} & J - 1 \\
2\pi r^n & 1 - r
\end{array} \right) \left\{ \begin{array}{ll}
S_1(z) & q=1 \\
S_2(z) & q \geq 2, k \geq 8q - 8/\beta - 2
\end{array} \right. \]

Schwarz inequality together with subordination for starlike functions [12] and (2.2) gives us, for $\beta(k+1)/2 - 1 \geq 0$,

\[ |\Delta_j(n, z_1, F)| \leq C(k, \beta, J)M^{1-\beta}(r) \left( \begin{array}{ll}
\frac{k+2}{2} & - 1 \\
1 - r & 1 - r
\end{array} \right) \]

where $C(k, \beta, J)$ is a constant which depends upon $k$, $\beta$ and $J$ only.

Applying lemma 2.5 and putting $z_1 = \left( \frac{n}{n+1} \right) e^{i\theta}$, we have for $k \geq 41/\beta - 2$

\[ \Delta_j(n, e^{i\theta}, f) = 0(1)M^{1-\beta}(r) n^{\beta(k+1)/2 - 1 - J - 1}, \]

where $0(1)$ depends on $k$, $\beta$ and $J$ only.

We now estimate the rate of growth of $H_q(n)$

For $q=1$, $H_q(n) = a_n = \Delta_0(n)$ and
For \( q \geq 2 \), we use the Remark due to Noonan and Thomas in [10] and we have

\[
H_q(n) = O(1) M^1 \beta \left( \frac{k}{2} + 1 \right) q^{-1}, \quad q \geq 2, \quad 0 < \beta \leq 1,
\]

and \( k > \frac{8(q-1)}{\beta} - 2 \). This completes the proof.

**Corollary 3.5**

When \( \beta = 1 \), \( f \in T_k \) and

\[
H_q(n) = O(1) \left( \frac{k}{2} + 1 \right) q^{-1}, \quad q \geq 2, \quad k \geq 8q-10.
\]

This result is known [13].

**Definition 3.1**

A function \( f \) is called \( \alpha \)-convex if, for \( \alpha > 0 \),

\[
\Re \left\{ \left( 1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)' \right\} > 0, \quad z \in \mathbb{E}.
\]

We now prove the following

**Theorem 3.5.**

Let \( f \in B_k(\beta) \), \( k \geq 2 \) and \( \beta \geq 1 \). Then \( f \) is \( \frac{1}{\beta} \)-convex for \( |z| < r_0 \), where

\[
r_0 = \frac{1}{2\beta} \left( (k\beta + 2) - \sqrt{(k\beta + 2)^2 - 4\beta^2} \right)
\]

**Proof:** Since \( f \in B_k(\beta) \), we have

\[
zf'(z) = f^{1-\beta} g^{\beta} h(z); \quad \Re h(z) > 0, \quad g \in R_k, \quad \text{from which it follows that}
\]

\[
\frac{1}{\beta} \left( \frac{zf'(z)}{f(z)} \right)' + \left( 1 - \frac{1}{\beta} \right) \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{z}{\beta} \frac{zh'(z)}{h(z)}, \quad (\beta \geq 1)
\]

Thus

\[
\Re \left[ \left( 1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)' \right] \geq \Re \frac{zg'(z)}{g(z)} - \frac{1}{\beta} \frac{zh'(z)}{h(z)}
\]

Since \( zg' = g \in R_k \) implies \( g \in V_k \), we have from a known result [14],

\[
\Re \frac{zg'(z)}{g(z)} = \Re \left( \frac{zg'(z)}{g(z)} \right)' > \frac{1-kr+r^2}{1-r^2},
\]

and for functions \( h \) of positive real part, it is known [15] that

\[
\frac{zh'(z)}{h(z)} < \frac{2r}{1-r^2}
\]

Using (3.2) and (3.3), we have

\[
\Re \left[ \left( 1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)' \right] > \frac{\beta(1-kr+r^2)-2r}{\beta(1-r^2)}
\]

and this gives us the required result.

**Corollary 3.6**

1. For \( k=2 \), \( \beta \geq 1 \), \( f \in B(\beta) \) is \( \frac{1}{\beta} \)-convex for \( |z| < r_1 = \frac{(\beta+1) - \sqrt{2\beta+1}}{\beta} \).
(ii) $\beta = 1$ implies $f \in T_k$ and it is convex for $|z| < r_2 = \frac{1}{2} \left( \frac{k+2}{\sqrt{k^2+4k}} \right)$.

This result is known, see [6].

(iii) When $k=2$, and $\beta = 1$, $f \in K$ and it is convex for $|z| < 2 - \sqrt{3}$.

**THEOREM 3.6.**

Let $G \in R_k$, $k \geq 2$. Let, for any positive integer, $\frac{1}{\alpha} = 2, 3, \ldots, n$ be defined as

$$h(z) = \int_0^z \frac{1}{t^\alpha} - 2 - \beta \frac{G(t)}{G(z)} \, dt$$

Then $h$ is $(\frac{1}{\alpha} - 1 + \beta)$ - valently starlike for $|z| < r_0$, where

$$r_0 = \frac{1}{2} (k-\sqrt{k^2-4}) \quad (3.4)$$

**PROOF:**

$$\left( \frac{zh'(z)}{h(z)} \right)' = \frac{1}{\alpha} - 1 + \beta \frac{G'(z)}{G(z)}$$

Since $G \in R_k$, it is known [16] that $\Re \frac{G'(z)}{G(z)} > 0$ for $|z| < r_0 = \frac{1}{2} (k-\sqrt{k^2-4})$.

Hence $h$ is convex and thus starlike in $|z| < r_0$. The $(\frac{1}{\alpha} - 1 + \beta)$ valency follows from the argument principle.

**THEOREM 3.7.**

Let $G \in R_k$, $k \geq 2$ and

$$g(z) = \frac{1}{\alpha} \int_0^z \left( \frac{1}{t^\alpha} - 2 - \beta \frac{G(t)}{G(z)} \right) \, dt,$$ \quad \text{where $\alpha$ and $\beta$ are defined as in theorem 3.6.} Then $g$ is starlike for $|z| < r_0$, where $r_0$ is given by (3.4).

**PROOF:**

$$g'(z) = \frac{\int_0^z \frac{1}{t^\alpha} - 2 - \beta \frac{G(t)}{G(z)} \, dt + \frac{1}{\alpha} - 1 - \frac{1}{\alpha} G'(z)}{D(z)}$$

where $D(z) = \int_0^z \frac{1}{t^\alpha} - 2 - \beta \frac{G(t)}{G(z)} \, dt$, which is $(\beta + \frac{1}{\alpha} - 1)$ - valently starlike for $|z| < r_0$, $r_0$ given by (3.4) from theorem 3.6.

Now

$$N'(z) = \beta \frac{zG'(z)}{G(z)}$$

and, for $|z| < r_0$, $r_0$ given by (3.4), we have

$$\Re \frac{N'(z)}{D'(z)} = \beta \Re \frac{zG'(z)}{G(z)} > 0,$$ \quad \text{since $G \in R_k$.}

Thus, using lemma 2.6, for $|z| < r_0$, it follows that

$$\beta \frac{zG'(z)}{G(z)} = \Re \frac{N(z)}{D(z)} > 0,$$ \quad \text{and this completes the proof.}

**Corollary 3.7.**

(i) For $k=2$, $\beta = 1$, we obtain Bernardi's result [17] for starlike functions.

(ii) Also, for $k=2$, $\alpha = 1/2$, we obtain a result proved in [18].
**Theorem 3.8.**

Let $f \in B_k(\beta)$, $z \in \mathbb{E}$ and let

$$f^\beta(z) = \frac{1}{\alpha} z^{1-1/\alpha} \int_0^z \left( \frac{1}{\alpha} - 2 \right) F^\beta(t) \, dt$$

where $\alpha$ and $\beta$ defined (3.5) as in theorem 3.6. Then $f \in B(\beta)$ for $|z| < r_0$, $r_0$ is given by (3.4).

**Proof:** Let $G \in R_k$ and let $g$ be defined as in theorem 3.7. Then $g$ is starlike for $|z| < r_0$, where $r_0$ is given by (3.4).

Now, from (3.5), we obtain

$$\frac{z f'(z)}{f^\beta(z) g}(z) = \frac{(1 - \frac{1}{\alpha}) \int_0^z \frac{1}{\alpha} - 2 F^\beta(t) \, dt + \frac{1}{\alpha} - 1 F^\beta(z)}{N(z) D(z)}$$

where $D(z) = \int_0^z \frac{1}{\alpha} - 2 G^\beta(t) \, dt$ is $(\beta + \frac{1}{\alpha} - 1)$ valently starlike for $|z| < r_0$. Also

$$\frac{N(z)}{D(z)} = \frac{z f'(z)}{f^\beta(z) g}(z) > 0,$$

since $f \in B_k(\beta)$.

Thus, using lemma 2.6, we obtain the desired result that $f \in B(\beta)$ for $|z| < r_0$, where $r_0$ is given by (3.4).

**Corollary 3.8.**

(i) For $k = 2$, $f \in B(\beta)$, $z \in \mathbb{E}$ and it follows from theorem 3.8 that $f \in B(\beta)$ for $|z| < 1$.

(ii) For $k = 2$, $\beta = 1$ implies $f \in K$ and from theorem 3.8 it follows that $f$ also belongs to $K$ for $|z| < 1$.

(iii) Let $\beta = 1$, then $f \in T_k$, and it follows from theorem 3.8 that $f$ is close-to-convex for $|z| < r_0$, given by (3.4). This is a generalization of a result proved in [13] for $\alpha = \frac{1}{2}$.

**Theorem 3.9.**

Let $f \in B_k(\beta)$. Then for $z = re^{i\theta}$, $0 < \theta_1 < \theta_2 < 2\pi$, $f(z) \neq 0$, $f'(z) \neq 0$ in $E$ and $0 < r < 1$, we have

$$\int_{\theta_1}^{\theta_2} \left\{ \frac{zf'(z)}{f(z)} + (\beta - 1) \frac{zf'(z)}{f(z)} \right\} d\theta = \frac{1}{2} \beta k \pi.$$

**Proof:** Since $f \in B_k(\beta)$ we can write

$$zf'(z) = f^\beta(z) g^\beta(z) h(z), \quad \text{Re} \, h(z) > 0, \quad g \in R_k.$$

Therefore,

$$\frac{zf'(z)}{f'(z)} + (\beta - 1) \frac{zf'(z)}{f(z)} = \beta \frac{(zT'(z))'}{T'(z)}. \quad (3.5)$$
Using a known result [6] for the class $T_k$, we have by integrating both sides of (3.5) between $\theta_1$, $\theta_2$, $0 < \theta_1 < \theta_2 < 2\pi$

$$\int_{\theta_1}^{\theta_2} \text{Re} \left( \frac{zf'(z)}{f'(z)} + (\beta-1) \frac{zf'(z)}{f(z)} \right) d\theta - \frac{\beta}{2} k \pi .$$

The following theorem shows the relationship between the classes $B_k(\beta)$ and $B(\beta)$. More precisely it gives the necessary condition for $f \in B_k(\beta)$ to be univalent.

**THEOREM 3.10**

Let $f \in B_k(\beta)$. Then $f$ is univalent if $k \leq \frac{2}{\beta}$, where $0 < \beta < 1$.

**PROOF:** The proof follows immediately from Theorem 3.9 and the result of Sheil-Small [19].

**REFERENCES**