ITERATIONS CONVERGING TO DISTINCT SOLUTIONS OF SOME NONLINEAR OPERATOR EQUATIONS IN BANACH SPACE

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ABSTRACT. We examine the solvability of multilinear equations of the form

\[ M_k(x,x,...,x) = y, \quad k = 2,3,... \]

where \( M_k \) is a \( k \)-linear operator on a Banach space \( X \) and \( y \in X \) is fixed.

KEY WORDS AND PHRASES. Multilinear operator, contraction.

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1. INTRODUCTION.

We study the quadratic equation

\[ B(x,x) y = \quad \]  

in a Banach space \( X \), where \( B \) is a bounded symmetric bilinear operator on \( X \) and \( y \) is fixed in \( X \) \([2],[3],[7],[9],[10]\). We consider two cases.

CASE 1. Let \( y = 0 \) and set \( x = \bar{x} - h \) for some \( \bar{x} \) such that the linear operator \( 2B(\bar{x}) \) is invertible then (1.1) becomes

\[ \vec{B}(h,h) = h - \bar{y} \]

where \( \vec{B} = (2B(\bar{x}))^{-1}B, \bar{y} = (2B(\bar{x}))^{-1}B(\bar{x},\bar{x}) \) and \( h \in X \) is to be determined.

We introduce the iteration

\[ h_{n+1} = (\vec{B}(h_n))^{-1}(h_n - \bar{y}) \quad \text{for some} \quad h_0 \in X \]

(1.3)

to find a solution \( h \) of (1.2) such that \( h \neq \bar{x} \).

It turns out under certain assumptions that iteration (1.3) converges to an \( h \in X \) such that \( h \neq \bar{x} \), therefore \( x = \bar{x} - h \) is a nonzero solution of (1.1).

CASE 2. Let \( y \neq 0 \), we then introduce the iteration

\[ x_{n+1} = B(x_n)^{-1}(y) \quad \text{for some} \quad x_0 \in X \]

(1.4)

to find solutions of (1.1).

The results obtained here can be generalized to include multilinear equations of the form
M_k(x, x, ..., x) = y
- k times -

where $M_k$ is a k-linear operator on X and y is fixed in X [10].

We now state the following lemma. The proof can be found in [10].

2. EXISTENCE THEORY.

**LEMMA 1.** Let $L_1$ and $L_2$ be bounded linear operators in a Banach space X, where $L_1$ is invertible, and $\|L_1^{-1}\| \cdot \|L_2\| < 1$. Then $(L_1 + L_2)^{-1}$ exists, and

$$\|L_1 + L_2\|^{-1} \leq \frac{\|L_1^{-1}\|}{1 - \|L_2\| \cdot \|L_1^{-1}\|}.$$  

**LEMMA 2.** Let $z \neq 0$ be fixed in X. Assume that the linear operator $\overline{B}(z)$ is invertible then $\overline{B}(x)$ is also invertible for all $x \in U(z, r) = \{x \in X | \|x-z\| < r\}$, where $r \in (0, r_0)$ and $r_0 = (\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|)^{-1}$.

**PROOF.** We have

$$\|\overline{B}(x-z)\| \cdot \|\overline{B}(z)^{-1}\| \leq \|\overline{B}\| \cdot \|x-z\| \cdot \|\overline{B}(z)^{-1}\|$$

$$\leq \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\| \cdot r$$

$$< 1$$

for $r \in (0, r_0)$. The result now follows from Lemma 1 for $L_1 = \overline{B}(z)$, $L_2 = \overline{B}(x-z)$ and $x \in U(z, r)$.

**DEFINITION 1.** Assume that the linear operator $\overline{B}(z)$ is invertible.

Define the operators $P, T$ on $U(z, r)$ for some $r > 0$ by

$$P(x) = \overline{B}(x, x) + \overline{y} - x, T(x) = (\overline{B}(x))^{-1}(x-\overline{y})$$

and the real polynomials $f(r), g(r)$ on $R$ by

$$f(r) = a'r^2 + b'r + c', \quad g(r) = ar^2 + br + c,$$

$$a' = (\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|)^2,$$

$$b' = -2\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|,$$

$$c' = 1 - \|\overline{B}(z)^{-1}\| - \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|^2 \cdot \|z-\overline{y}\|,$$

$$a = \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|,$$

$$b = \|\overline{B}(z)^{-1}(I-\overline{B}(z))\| - 1,$$

and

$$c = \|\overline{B}(z)^{-1}P(z)\|.$$

**THEOREM 1.** Let $z \in X$ be such that $\overline{B}(z)$ is invertible and that the following are true:

a) $c' > 0$;

b) $b < 0, b^2 - 4ac > 0$, and

c) there exists $r > 0$ such that $f(r) > 0$ and $f(r) \leq 0$

then the iteration

$$h_{n+1} = \overline{B}(h_n)^{-1}(h_n - \overline{y}), \quad n = 0, 1, 2, ...$$
is well defined and it converges to a unique solution $h$ of (1.2) in $\overline{U}(z,r)$ for any $h_0 \in \overline{U}(z,r)$.

**PROOF.** $T$ is well defined by Lemma 2.

**CLAIM 1.** $T$ maps $\overline{U}(z,r)$ into $\overline{U}(z,r)$.

If $x \in \overline{U}(z,r)$ then

$$T(x) - z = \overline{E}(x)^{-1}(x-y) - z$$

$$= \overline{E}(x)^{-1}[(1 - \overline{E}(z))(x-z) - F(z)]$$

so

$$\|T(x) - z\| \leq r$$

if

$$\frac{1}{1 - \|E\| \cdot \|\overline{E}(z)^{-1}\| \cdot r} ([\|\overline{E}(z)^{-1}(1 - \overline{E}(z))\| \cdot r + \|\overline{E}(z)^{-1}F(z)\|] \leq r$$

(uses Lemma 1 for $L_1 = \overline{E}(z)$ and $L_2 = \overline{E}(x-z)$) or $g(r) \leq 0$ which is true by hypothesis.

**CLAIM 2.** $T$ is a contraction operator on $\overline{U}(z,r)$.

If $w, v \in \overline{U}(z,r)$ then

$$\|T(w) - T(v)\|$$

$$= \|\overline{E}(w)^{-1}(w-y) - \overline{E}(v)^{-1}(v-y)\|$$

$$= \|\overline{E}(w)^{-1}[1 - \overline{E}(\overline{E}(v)^{-1}(v-y))](w-v)\|$$

$$\leq \frac{1}{1 - \|E\| \cdot \|\overline{E}(z)^{-1}\| \cdot r} \left[ \|\overline{E}(z)^{-1}\| + \|E\| \cdot \|\overline{E}(z)^{-1}\|^2 \cdot r + \|E\| \cdot \|\overline{E}(z)^{-1}\|^2 \cdot z \cdot \overline{E}\| \right] \cdot \|w-v\|$$

$$= q \cdot \|w-v\|.$$

So $T$ is a contraction on $\overline{U}(z,r)$ if $0 < q < 1$, where

$$q = \frac{1}{1 - \|E\| \cdot \|\overline{E}(z)^{-1}\| \cdot r} \left[ \|\overline{E}(z)^{-1}\| + \|E\| \cdot \|\overline{E}(z)^{-1}\|^2 \cdot r + \|E\| \cdot \|\overline{E}(z)^{-1}\|^2 \cdot z \cdot \overline{E}\| \right] \cdot \|w-v\|$$

which is true since $f(r) > 0$.

**THEOREM 2.** Assume that there exist $r > 0$, $z, \overline{x} \in X$ satisfying the hypotheses of Theorem 1 and

(a) $0 < \|\overline{x}\| < -1 + \frac{\sqrt{1 + 4\|E\| \cdot \|\overline{E}\|}}{2\|\overline{E}\|}$;

(b) $r + \|z\| < \frac{\|\overline{y}\|}{1 + \|\overline{E}\| \cdot \|\overline{x}\|}$

then if $\|\overline{x}\| < h_0 \leq r + \|z\|$, the solution $h$ if (1.2) is such that

$\|\overline{x}\| < \|h\| \leq r + \|z\|$.

Moreover, $x = \overline{x} - h$ is a nonzero solution of (1.1).

**PROOF.** By Theorem 1 $h \in \overline{U}(z,r)$ therefore

$\|h\| \leq r + \|z\|$.
Assume that \( \|h_k\| > \|x\| \) for \( k = 0, 1, 2, \ldots, n \). By iteration (1.3) we have
\[
B(h_{n+1}, h_n) = h_n - \bar{y}
\]
or
\[
\|B\| \|h_{n+1}\| \cdot \|h_n\| \geq \|h_n - \bar{y}\| \geq \|\bar{y}\| - \|h_n\|
\]
so
\[
\|h_{n+1}\| \geq \frac{\|\bar{y}\| - \|h_n\|}{\|B\| \cdot \|h_n\|},
\]
to show that
\[
\|h_{n+1}\| > \|x\|,
\]
it suffices to show
\[
\frac{\|\bar{y}\| - \|h_n\|}{\|B\| \cdot \|h_n\|} > \|x\|
\]
which is true by (b). For consistency we must have
\[
\|x\| < \frac{\|\bar{y}\|}{1 + \|B\| \cdot \|h_n\|}
\]
which is true by (a). The result now follows by taking the limit as \( n \to \infty \) in (2.1).

Finally note that since \( \|h\| > \|x\| \), \( x - h \neq 0 \) therefore \( x = \bar{x} - h \) is a non-zero solution of (1.1).

**DEFINITION 2.** Assume that the linear operator \( B(z) \) is invertible for some \( z \in X \). Define the operator \( \bar{P} \) on \( U(z, r) \) for some \( r > 0 \) by
\[
\bar{P}(x) = B(x, x) - y, \; y \neq 0
\]
and the real polynomials \( \bar{f}(r), \bar{g}(r) \) on \( R \) by
\[
\bar{f}(r) = s_1 r^2 + s_2 r + s_3, \; \bar{g}(r) = s_1 r - s_2 r + s_3,
\]
where
\[
\begin{align*}
&s_1' = (\|B\| \cdot \|B(z)^{-1}\|)^2 \\
&s_2' = -2\|B\| \cdot \|B(z)^{-1}\| \\
&s_3' = 1 - \|B\| \cdot \|B(z)^{-1}\| \\
&s_1 = \|B\| \cdot \|B(z)^{-1}\| \\
&s_2 = \|B\| \\
&s_3 = \|B(z)^{-1}\|.
\end{align*}
\]

The proofs of the following theorems are omitted as similar to Theorems 1 and 2.

**THEOREM 3.** Let \( z \in X \) be such that the linear operator \( B(z) \) is invertible and that the following are true:

a) \( s_3' > 0 \);

b) \( s_2 > 0, s_2' - \frac{s_1 s_2}{s_3} > 0 \), and

c) there exists \( r > 0 \) such that \( \bar{f}(r) > 0 \) and \( \bar{g}(r) \leq 0 \).
then the iteration
\[ x_{n+1} = B(x_n)^{-1}(y) \]
for some \( x_0 \in X \) is well defined and it converges to a solution \( x \) of (1.1) which is unique in \( 
U(z,r) \) for any \( x_0 \in U(z,r) \).

**THEOREM 4.** Let \( z, r \) be such that the hypotheses of Theorem 3 are satisfied. Let \( p < q \) be positive numbers such that
\[ p \|B\| \leq \|y\| ; \]
\[ \frac{\|B(z)^{-1}\|}{1 - \|B\| \cdot \|B(z)^{-1}\|r} \leq q \leq r + \|z\| \]
then if \( p \leq \|x_0\| \leq q \) then the solution \( x \) of (1.1) is such that
\[ p \leq \|x\| \leq q. \]

**REFERENCES**


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