Sunlet Decomposition of Certain Equipartite Graphs

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1. Introduction

Let \( C_r, K_n, \overline{K}_m \) denote cycle of length \( r \), complete graph on \( n \) vertices, and complement of complete graph on \( m \) vertices. For \( n \) even, \( K_n + I \) denotes the multigraph obtained by adding the edges of a 1-factor to \( K_n \), thus duplicating \( n/2 \) edges. The total number of edges in \( K_n + I \) is \( n^2/2 \). The lexicographic product, \( G \circ H \), of graphs \( G \) and \( H \), is the graph obtained by replacing every vertex of \( G \) by a copy of \( H \) and every edge of \( G \) by the complete bipartite graph \( K_{|G|,|H|} \).

For a graph \( H \), an \( H \)-decomposition of a graph \( G, H \mid G \), is a set of subgraphs of \( G \), each isomorphic to \( H \), whose edge set partitions the edge set of \( G \). Note that for any graph \( G \) and \( H \) and any positive integer \( m \), if \( H \mid G \) then \((H \circ \overline{K}_m) \mid (G \circ \overline{K}_m) \).

Let \( G \) be a graph of order \( n \) and \( H \) any graph. The corona (crown) of \( G \) with \( H \), denoted by \( G \odot H \), is the graph obtained by taking one copy of \( G \) and \( n \) copies of \( H \) and joining the \( i \)th vertex of \( G \) with every vertex in the \( i \)th copy of \( H \). A special corona graph is \( C_r \odot K_1 \), that is, a cycle with pendant points which has \( 2n \) vertices. This is called sunlet graph and denoted by \( L_{r,q} \), where \( q = 2n \).

Obvious necessary condition for the existence of a \( k \)-cycle decomposition of a simple connected graph \( G \) is that \( G \) has at least \( k \) vertices (or trivially, just one vertex), the degree of every vertex in \( G \) is even, and the total number of edges in \( G \) is a multiple of the cycle length \( k \). These conditions have been shown to be sufficient in the case that \( G \) is the complete graph \( K_n \), the complete graph minus a 1-factor \( K_n - I \) [1, 2], and the complete graph plus a 1-factor \( K_n + I \) [3].

The study of cycle decomposition of \( K_n \circ \overline{K}_m \) was initiated by Hoffman et al. [4]. The necessary and sufficient conditions for the existence of a \( C_p \)-decomposition of \( K_n \circ \overline{K}_m \), where \( p \geq 5 \) (\( p \) is prime) that (i) \( m(n - 1) \) is even and (ii) \( p \) divides \( n(n - 1)m^2 \), were obtained by Manikandan and Paulraja [5, 6]. Similarly, when \( p \geq 3 \) is a prime, the necessary and sufficient conditions for the existence of a \( C_{2p} \)-decomposition of \( K_n \circ \overline{K}_m \) were given by Smith [7]. For a prime number \( p \geq 3 \), Smith [8] showed that \( C_{3p} \)-decomposition of \( K_n \circ \overline{K}_m \) exists if the obvious necessary conditions are satisfied. In [9], Anitha and Lekshmi proved that the complete graph \( K_n \) and the complete bipartite graph \( K_n, n \) for \( n \) even have decompositions into sunlet graph \( L_n \). Similarly, in [10], it was shown that the complete equipartite graph \( K_n \circ \overline{K}_m \) has a decomposition into sunlet graph of length \( 2p \), for a prime \( p \).

We extend these results by considering the decomposition of \( K_n + I \circ \overline{K}_m \) into sunlet graph and prove the following result.

Let \( m \geq 2, n > 2, \) and \( q \geq 6 \) be even integers. The graph \( K_n + I \circ \overline{K}_m \) can be decomposed into sunlet graph of length \( q \) if and only if \( q \) divides \( n^2m^2/2 \), the number of edges in \( K_n + I \circ \overline{K}_m \).
2. Proof of the Result

To prove the result, we need the following.

Lemma 1 (see [10]). For $r \geq 3$, $L_n$ decomposes $C_r \ast K_2$.

Lemma 2. For any integer $r > 2$ and a positive even integer $m$, the graph $C_r \ast \overline{K}_m$ has a decomposition into sunlet graph $L_q$, for $q = rm$.

Proof

Case 1 ($r$ is even). First observe that $C_r \ast \overline{K}_2$ can be decomposed into 2 sunlet graphs with 2r vertices. Now, set $m = 2t$ and decompose $C_r \ast \overline{K}_2$ into cycles $C_{rt}$, to decompose $C_r \ast \overline{K}_2$ into t-cycles $C_{rt}$, denote vertices in $i$th part of $C_r \ast \overline{K}_2$ by $x_{ij}$, for $j = 1, \ldots, t$, $i = 1, 2, \ldots, r$ and create t base cycles $x_1x_2x_3 \cdots x_r$. Next, combine these base cycles into one cycle $C_{rt}$, by replacing each edge $x_{i}x_{i+1}$ with $x_i,x_{i+2}$. To create the remaining cycles $C_{rt}$, we apply mappings $\phi_i$ for $s = 0, 1, \ldots, t - 1$ defined on the vertices as follows.

Subcase 1.1 (i odd). Consider

$$\phi_i(x_{ij}) = x_{ij}.$$  

This is the desired decomposition into cycles $C_{rt}$.

Subcase 1.2 (i even). Consider

$$\phi_i(x_{ij}) = x_{i+1j},$$  

This is the desired decomposition into cycles $C_{rt}$.

Now take each cycle $C_{rt}$, and make it back into $C_r \ast \overline{K}_2$. Each $C_r \ast \overline{K}_2$ decomposes into 2 sunlet graphs $L_{2t}$ (by Lemma 1), and we have $C_r \ast \overline{K}_m$ decomposing into sunlet graphs with length $rm$ for $r$ even. Note that

$$C_r \ast \overline{K}_{2t} = (C_r \ast \overline{K}_t) \ast \overline{K}_2.$$  

Case 2 ($r$ is odd)

Subcase 2.1 ($m \equiv 2$ (mod 4)). Set $m = 2t$. First create $t$ cycles $C_{(r-1)t}$ in $C_{(r-1)t} \ast \overline{K}_2$, as in Case 1. Then, take complete tripartite graph $K_{tj}$ with partite sets $X_i = \{x_{ij}\}$ for $i = 1, r-1, r$ and $j = 1, \ldots, t$ and decompose it into triangles using well-known construction via Latin square, that is, construct $t \times t$ Latin square and consider each element in the form $(a, b, c)$ where $a$ denotes the row, $b$ denotes the column, and $c$ denotes the entry with $1 \leq a, b, c \leq t$. Each cycle is of the form $x_{11}x_{21}x_{31}x_{41} \cdots x_{ri}$, where, for every triangle $x_{1a}x_{1b}x_{1c}, x_{2a}x_{2b}x_{2c}, \ldots, x_{ri}x_{ri+1}x_{ri+2}$, replace the edge $x_{1a}x_{1b}$ in each $C_{(r-1)t}$, by the edges $x_{2a}x_{2b}$ and $x_{2c}x_{ri}$, to obtain cycles $C_{rt}$. Therefore, $C_{rt} \ast \overline{K}_2$. Now take each cycle $C_{rt}$, make it into $C_r \ast \overline{K}_2$, and by Lemma 1, $C_r \ast \overline{K}_2$ has a decomposition into sunlet graphs $L_{2rt} = L_q$.

Subcase 2.2 ($m \equiv 0$ (mod 4)). Set $m = 2t$. The graph $C_r \ast \overline{K}_2$ decomposes into Hamilton cycle $C_{rt}$ by [11]. Next, make each cycle $C_{rt}$ into $C_r \ast \overline{K}_2$. Each graph $C_r \ast \overline{K}_2$ decomposes into sunlet graph $L_{2rt}$ by Lemma 1.

Theorem 3. Let $r$, $m$ be positive integers satisfying $r, m \equiv 0$ (mod 4), then $L_r$ decomposes $C_r \ast \overline{K}_m$.

Proof. Let the partite sets (layers) of the $r$-partite graph $C_r \ast \overline{K}_m$ be $U_1, U_2, \ldots, U_r$. Set $m = 2t$. Obtain a new graph from $C_r \ast \overline{K}_m$ as follows.

Identify the subsets of vertices $\{x_{ij}\}$, for $1 \leq i \leq r$ and $1 \leq j \leq m/2$ into new vertices $x_{ij}^1$, and identify the subset of vertices $\{x_{ij}\}$ for $1 \leq i \leq r$ and $m/2 + 1 \leq j \leq m$ into new vertices $x_{ij}^2$, and two of these vertices $x_{ij}^k$, where $k = 1, 2$, are adjacent if and only if the corresponding subsets of vertices in $C_r \ast \overline{K}_m$ induce $K_{r,t}$. The resulting graph is isomorphic to $C_r \ast \overline{K}_2$. Next, decompose $C_r \ast \overline{K}_2$ into cycles $C_{rt/2}$ as follows:

$$k = 1, \frac{r}{4} + 1, \frac{r}{2} + 1, \frac{3r}{4} + 1, \ldots, -\frac{r}{4} + 1, \quad d = \frac{r}{4} + k,$$

where $k, d$ are calculated modulo $r$.

To construct the remaining cycles, apply mapping $\phi$ defined on the vertices.

Subcase 1.1 (i odd in each cycle). Consider

$$\phi(x_{ij}) = x_{i,j+1}.$$  

This is the desired decomposition of $C_r \ast \overline{K}_2$ into cycles $C_{rt/2}$.

Subcase 1.2 (i even in each cycle). Consider

$$\phi(x_{ij}) = x_{i,j}.$$  

This is the desired decomposition of $C_r \ast \overline{K}_2$ into cycles $C_{rt/2}$.

By lifting back these cycles $C_{rt/2}$ of $C_r \ast \overline{K}_2$ to $C_r \ast \overline{K}_m$, we get edge-disjoint subgraphs isomorphic to $C_{rt/2} \ast \overline{K}_1$. Obtain a new graph again from $C_{rt/2} \ast \overline{K}_m$ as follows.

For each $j$, $1 \leq j \leq t/2$, identify the subsets of vertices $\{x_{(2j-1),1}, \ldots, x_{(2j-1),t}\}$, where $1 \leq i \leq r/2$ into new vertices $x_{ij}^1$, and two of these vertices $x_{ij}^1$ are adjacent if and only if the corresponding subsets of vertices in $C_{rt/2} \ast \overline{K}_m$ induce $K_{r,t/2}$. The resulting graph is isomorphic to $C_{rt/2} \ast \overline{K}_{t/2}$. Then, decompose $C_{rt/2} \ast \overline{K}_{t/2}$ into cycles $C_{rt/2}$. Each $C_{rt/2} \ast \overline{K}_{t/2}$ decomposes into cycles $C_{rt/2}$ by [12]. By lifting back these cycles $C_{rt/2}$ of $C_{rt/2} \ast \overline{K}_{t/2}$ to $C_{rt/2} \ast \overline{K}_1$, we get edge-disjoint subgraph isomorphic to $C_{rt/2} \ast \overline{K}_1$. Finally, each $C_{rt/2} \ast \overline{K}_1$ decomposes into two sunlet graphs $L_r$ (by Lemma 1), and we have $C_r \ast \overline{K}_m$ decomposing into sunlet graphs $L_r$, as required.  

Theorem 4 (see [12]). The cycle $C_m$ decomposes $C_k \ast \overline{K}_m$ for every even $m > 3$.

Theorem 5 (see [12]). If $m$ and $k \geq 3$ are odd integers, then $C_m$ decomposes $C_k \ast \overline{K}_m$.  

**Theorem 6.** The sunlet graph $L_m$ decomposes $C_r * \overline{K}_m$ if and only if either one of the following conditions is satisfied.

1. $r$ is a positive odd integer, and $m$ is a positive even integer.
2. $r$, $m$ are positive even integers with $m \equiv 0 \pmod{4}$.

**Proof.** (1) Set $m = 2t$, where $t$ is a positive integer. Let the partite sets (layers) of the $r$-partite graph $C_r * K_m$ be $U_1, U_2, \ldots, U_r$. For each $j$, where $1 \leq j \leq t$, identify the subsets of vertices $\{x_{1,2j-1}, x_{1,2j}\}$, for $1 \leq i \leq r$ into new vertices $x'_i$, and two of these vertices $x'_i$ are adjacent if and only if the corresponding subsets of vertices in $C_r * \overline{K}_m$ induce $K_2$. The resulting graph is isomorphic to $C_r * \overline{K}_r$. Then, decompose $C_r * \overline{K}_r$ into sunlet graphs, where $t$ is a positive integer.

Now, $C_r | C_r * \overline{K}_r$ by Theorems 4 and 5.

By lifting back these $t$-cycles of $C_r * \overline{K}_r$, we get edge-disjoint subgraphs isomorphic to $C_r * \overline{K}_2$. Each copy of $C_r * \overline{K}_2$ decomposes into sunlet graphs of length $2t$ (by Lemma 1), and we have $C_r * \overline{K}_r$ decomposing into sunlet graphs of length $m$ as required.

(2) Set $m = 2t$, where $t$ is an even integer since $m \equiv 0 \pmod{4}$.

Obtain a new graph $C_r * \overline{K}_r$ from the graph $C_r * \overline{K}_m$ as in Case 1. By Theorem 4, $C_r | C_r * \overline{K}_r$. By lifting back these $t$-cycles of $C_r * \overline{K}_r$ to $C_r * \overline{K}_2$, we get edge-disjoint subgraphs isomorphic to $C_r * \overline{K}_2$. Each copy of $C_r * \overline{K}_2$ decomposes into sunlet graphs of length $2t$ (by Lemma 1). Therefore, $L_m | C_r * \overline{K}_r$ as required. \(\square\)

**Remark 7.** In [10], it was shown that

$$L_{2r} * \overline{K}_1$$

can be decomposed into $l^2$ copies of $L_{2r}$. (7)

This, coupled with Lemma 1, gives the following.

**Theorem 8 (see [10]).** The graph $C_r * \overline{K}_m$ decomposes into sunlet graphs $L_{2r}$ for any positive integer $l$.

**Lemma 9** (see [3]). Let $n \geq 4$ be an even integer. Then, $K_n + I$ is $C_r$-decomposable.

**Lemma 10** (see [3]). Let $m$ and $n$ be integers with $m$ odd, $n \equiv 2 \pmod{4}$, $3 \leq m \leq n < 2m$, and $n^2 \equiv 0 \pmod{2m}$. Then, $K_n + I$ is $C_m$-decomposable.

**Lemma 11** (see [3]). Let $m$ and $n$ be integers with $m$ odd, $n \equiv 0 \pmod{4}$, $3 \leq m \leq n < 2m$, and $n^2 \equiv 0 \pmod{2m}$. Then, $K_n + I$ is $C_m$-decomposable.

We can now prove the major result.

**Theorem 12.** For any even integers $m \geq 2$, $n > 2$, and $q \geq 6$, the sunlet graph $L_q$ decomposes $K_n + I * \overline{K}_m$ if and only if $n^2m^2/2 \equiv 0 \pmod{q}$.
References


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