Research Article
Ramsey Numbers for Theta Graphs

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The graph Ramsey number \( R(F_1, F_2) \) is the smallest integer \( N \) with the property that any complete graph of at least \( N \) vertices whose edges are colored with two colors (say, red and blue) contains either a subgraph isomorphic to \( F_1 \) all of whose edges are red or a subgraph isomorphic to \( F_2 \) all of whose edges are blue. In this paper, we consider the Ramsey numbers for theta graphs. We determine \( R(\theta_n, \theta_k) \), \( R(\theta_n, 2\theta_k) \) for \( k \geq 4 \). More specifically, we establish that \( R(\theta_n, \theta_k) = R(2\theta_n, \theta_k) = 2k - 1 \) for \( k \geq 7 \). Furthermore, we determine \( R(\theta_n, \theta_n) \) for \( n \geq 5 \). In fact, we establish that \( R(\theta_n, \theta_n) = (3n/2) - 1 \) if \( n \) is even, \( 2n - 1 \) if \( n \) is odd.

1. Introduction and Preliminaries

The graphs considered in this paper are finite, undirected, and have no loops or multiple edges. For a given graph \( G \), we denote the vertex set of a graph \( G \) by \( V(G) \) and the edge set by \( E(G) \). The cardinalities of these sets are denoted by \( |V(G)| \) and \( |E(G)| \), respectively. Throughout this paper a cycle on \( m \) vertices will be denoted by \( C_m \), the complete graph on \( n \) vertices by \( K_n \). Suppose that \( V_1 \subseteq V(G) \) and \( V_1 \) is non-empty, the subgraph of \( G \) whose vertex set is \( V_1 \) and whose edge set is the set of those edges of \( G \) that have both ends in \( V_1 \) is called the subgraph of \( G \) induced by \( V_1 \), denoted by \( G[V_1] \). Let \( C \) be a cycle in a graph \( G \), an edge in \( E(G[C]) \backslash E(C) \) is called a chord of \( C \). Further, a graph \( G \) has a \( \theta_n \)-graph if \( G \) has a cycle \( C_n \) that has a chord in \( G \). Let \( G \) be a graph and \( u \in V(G) \). The degree of a vertex \( u \) in \( G \), denoted by \( d_G(u) \), is the number of edges of \( G \) incident to \( u \). The neighbor-set of a vertex \( u \) of \( G \) in a subgraph \( H \) of \( G \), denoted by \( N_H(u) \), consists of the vertices of \( H \) adjacent to \( u \). The circumference, \( c(G) \), of the graph \( G \) is the length of the longest cycle of \( G \). For vertex-disjoint subgraphs \( H_1 \) and \( H_2 \) of \( G \) we let \( E(H_1, H_2) = \{ xy \in E(G) : x \in V(H_1), \ y \in V(H_2) \} \) and \( E(H_1, H_2) = |E(H_1, H_2)| \).
The graph Ramsey number \( R(F_1, F_2) \) is the smallest integer \( N \) with the property that any complete graph of at least \( N \) vertices whose edges are colored with two colors (say, red and blue) contains either a subgraph isomorphic to \( F_1 \) all of whose edges are red or a subgraph isomorphic to \( F_2 \) all of whose edges are blue.

It is well known that the problem of determining the Ramsey numbers for complete graphs is very difficult, and it is easier to deal with paths, trees, cycles, and theta graphs. See the updated bibliography by Radziszowski [1]. In this paper we study \( R(F_1, F_2) \) in the case when both \( F_1 \) and \( F_2 \) are theta graphs.

The results concerning Ramsey numbers for cycles were established by Chartrand and Schuster [2] (for \( k < 7 \)), by Bondy and Erdős [3] (for \( n = k \) odd and for the case when \( k \) is much smaller than \( n \)), and for all the remaining values by Rosta [4] and by Faudree and Schelp [5], independently. These results are summarized in the following theorem.

**Theorem 1.1.** Let \( 3 \leq m \leq n \) be integers. Then

\[
R(C_n, C_m) = \begin{cases} 
6 & \text{if } m = n = 3 \text{ or } 4, \\
 n + \frac{m}{2} - 1 & \text{if } n, m \text{ are even, } (n, m) \neq (4, 4), \\
\max\left\{ n + \frac{m}{2} - 1, 2m - 1 \right\} & \text{if } n \text{ is odd, } m \text{ is even,} \\
2n - 1 & \text{if } m \text{ is odd, } (n, m) \neq (3, 3).
\end{cases}
\] (1.1)

In order to prove our results, we need to state the following results.

**Theorem 1.2** (see [6]). Let \( G \) be a graph on \( n \) vertices with no cycles of length greater than \( k \). Then \( \mathcal{E}(G) \leq (1/2)k(n - 1) - (1/2)r(k - r - 1) \) where \( r = (n - 1) - (k - 1)(n - 1)/(k - 1) \).

**Theorem 1.3** (see [7]). Every non-bipartite graph \( G \) on \( n \) vertices with more than \( \left\lfloor (n - 1)^2/4 \right\rfloor + 1 \) edges contains cycles of every length \( l \), where \( 3 \leq l \leq c(G) \).

**Theorem 1.4** (see [8]). Let \( G \) be a non-bipartite graph on \( n \geq 7 \) vertices and \( G \) contains no \( \theta_4 \)-subgraph. Then \( \mathcal{E}(G) \leq \left\lfloor (n - 1)^2/4 \right\rfloor + 2 \).

**Theorem 1.5** (see [9]). Let \( G \) be a non-bipartite graph on \( n \geq 9 \) vertices and \( G \) contains no \( \theta_5 \)-subgraph. Then \( \mathcal{E}(G) \leq \left\lfloor (n - 1)^2/4 \right\rfloor + 1 \).

In this paper, we consider Ramsey numbers for theta graphs. We determine \( R(\theta_4, \theta_k) \), \( R(\theta_5, \theta_k) \) for \( k \geq 4 \). More specifically, we establish that \( R(\theta_4, \theta_k) = R(\theta_5, \theta_k) = 2k - 1 \) for \( k \geq 7 \). Furthermore, we determine \( R(\theta_n, \theta_n) \) for \( n \geq 5 \). In fact, we establish that

\[
R(\theta_n, \theta_n) = \begin{cases} 
\left( \frac{3n}{2} \right) - 1 & \text{if } n \text{ is even,} \\
2n - 1 & \text{if } n \text{ is odd.}
\end{cases}
\] (1.2)

Throughout this paper (Figures 1–5), solid lines represent red edges and dashed lines represent blue edges.
2. Main Result

In the following lemma we determine the Ramsey number \( R(\theta_4, C_3) \).

**Lemma 2.1.** The Ramsey number \( R(\theta_4, C_3) = 7 \).

**Proof.** First we show that \( R(\theta_4, C_3) \geq 7 \). Let \( K_6 \) be colored as follows: the vertex set \( V(K_6) \) is the disjoint union of two subsets \( H_1 \) and \( H_2 \) each of order 3 and completely colored red. All edges between \( H_1 \) and \( H_2 \) are colored blue. This coloring contains neither a red \( \theta_4 \)-graph nor a blue \( C_3 \). So, we conclude that \( R(\theta_4, C_3) \geq 7 \).

It remains to show that \( R(\theta_4, C_3) \leq 7 \). Let a red-blue coloring of \( K_7 \) be given. By Theorem 1.1, \( K_7 \) has a red \( C_3 \) or a blue \( C_3 \). If \( K_7 \) has a blue \( C_3 \), then the result is obtained. So we need to consider the case when \( K_7 \) has a red \( C_3 \). Let \( x_1, x_2, \ldots, x_7 \) be the vertices of \( K_7 \) and assume \( x_1, x_2, x_3 \) are the vertices of the red \( C_3 \). Any vertex of the remaining vertices \( x_4, x_5, x_6, \) and \( x_7 \) is adjacent to the red \( C_3 \) by at least 2 blue edges as otherwise a red \( \theta_4 \)-graph is produced. Assume \( K_7[x_4, x_5, x_6, x_7] \) has a blue edge, say \( x_4x_5 \), then \( K_7[x_1, x_2, x_3, x_4, x_5] \) has a blue \( C_3 \). So, we need to consider that \( K_7[x_4, x_5, x_6, x_7] \) has no blue edge. Thus, a red \( \theta_4 \)-graph is produced. This completes the proof. \( \Box \)

In the following lemma we determine the Ramsey number \( R(\theta_4, \theta_4) \).

**Lemma 2.2.** The Ramsey number \( R(\theta_4, \theta_4) = 10 \).

**Proof.** First we show that \( R(\theta_4, \theta_4) \geq 10 \). Let \( K_9 \) be colored as follows: The vertex set \( V(K_9) \) is the disjoint union of three subsets \( G_1, G_2 \), and \( G_3 \) each of order 3 and completely colored red and the red edges between \( G_1, G_2 \), and \( G_3 \) are shown in Figure 1. The remaining edges are colored blue.

This coloring contains neither a red nor a blue \( \theta_4 \)-graph. So, we conclude that \( R(\theta_4, \theta_4) \geq 10 \). It remains to show that \( R(\theta_4, \theta_4) \leq 10 \). Let a red-blue coloring of \( K_{10} \) be given. By Theorem 1.1, \( K_{10} \) contains a red or a blue \( C_3 \). Without loss of generality we assume that \( K_{10} \) has a red \( C_3 \). Let \( x_1, x_2, \ldots, x_{10} \) be the vertices of \( K_{10} \) and assume \( x_1x_2x_3x_4 \) be the red \( C_3 \) in \( K_{10} \). Observe that \( H_1 = K_{10}[x_4, x_5, \ldots, x_{10}] \) has a red \( C_3 \) or a blue \( C_3 \). So, we consider the following two cases.

**Case 1.** Suppose \( H_1 \) has a red \( C_3 \), say \( x_5x_6x_{10}x_5 \). Let \( x_4, x_5, x_6, \) and \( x_7 \) be the remaining vertices in \( K_{10} \). Suppose \( H_2 = K_{10}[x_4, x_5, x_6, x_7] \) has no blue edge, then \( K_{10} \) has a red \( \theta_4 \)-graph. So, we need to consider the case when \( H_2 \) has a blue edge, say \( x_4x_5 \) is the blue edge. Observe that any vertex in \( H_2 \) must be adjacent to each of \( K_{10}[x_1, x_2, x_3] \) and \( K_{10}[x_8, x_9, x_{10}] \) by two blue edges as otherwise a red \( \theta_4 \)-graph is produced. Thus \( x_4x_5 \) and \( x_5 \) incident with two blue edges that have a common vertex in \( K_{10}[x_1, x_2, x_3] \) and incident with two blue edges that have a common vertex in \( K_{10}[x_8, x_9, x_{10}] \) and so a blue \( \theta_4 \)-graph is produced.

**Case 2.** Now we need to consider the case when \( H_1 \) has a blue \( C_3 \), say \( x_5x_6x_{10}x_5 \). Observe that every vertex in the red \( C_3 \) is adjacent to the blue \( C_3 \) by two red edges as otherwise a blue \( \theta_4 \)-graph is produced. Further, every vertex in the blue \( C_3 \) is adjacent to the red \( C_3 \) by two blue edges as otherwise a red \( \theta_4 \)-graph is produced.

Thus, there are at least six red edges between the red \( C_3 \) and the blue \( C_3 \) and at least six blue edges between the blue \( C_3 \) and the red \( C_3 \). We know that, \( E(C_3, C_3) = 9 \). This is a contradiction. \( \Box \)
In the following lemma we determine the Ramsey number $R(\theta_5, \theta_5) = 9$.

**Lemma 2.3.** The Ramsey number $R(\theta_5, \theta_5) = 9$.

**Proof.** It is enough to show that $R(\theta_5, \theta_5) \leq 9$. Let a red-blue coloring of $K_9$ be given that contains neither a red $\theta_5$-graph nor a blue $\theta_3$-graph. By Theorem 1.1, $K_9$ must contain a blue $C_5$ or a red $C_5$. Without loss of generality we assume that $K_9$ has a red $C_5$. Let $x_1, x_2, \ldots, x_9$ be the vertices of $K_9$ and assume $x_1, x_2, x_3, x_5, x_1$ is the red $C_5$. Define $H_1 = K_9[x_1, x_2, x_3, x_4, x_5]$ and $H_2 = K_9[x_6, x_7, x_8, x_9]$. Now we have the following observations.

(i) $H_1$ has no red chord as otherwise a red $\theta_5$-graph is produced. Thus, $H_1$ contains a blue $C_5$.

(ii) Every vertex in $H_2$ is adjacent by at most 3 red (blue) edges to $H_1$ as otherwise a red (blue) $\theta_3$-graph is produced.

(iii) If a vertex in $H_2$ is adjacent to $H_1$ by 3 red (blue) edges, then it must be adjacent to 3 consecutive vertices in $H_1$ with red (blue) color as otherwise $K_9$ would have a red (blue) $\theta_5$-graph. Let $a, b, c$ be consecutive with red (blue) if $a$ is adjacent to $b$ with a red (blue) edge and $b$ is adjacent to $c$ with a red (blue) edge.

(iv) Assume there are two vertices in $H_2$, say $x_6$ and $x_7$ are adjacent to $H_1$ by 3 red (blue) edges each. Then $x_6$ and $x_7$ are adjacent to 3 consecutive vertices in $H_1$ and $|N_{H_1}(x_6) \cap N_{H_1}(x_7)| = 1$ as otherwise a red (blue) $\theta_3$-graph is produced.

(v) There are exactly two vertices in $H_2$ adjacent to vertices of $H_1$ by exactly 3 red edges and two blue edges each, say $x_6$ and $x_7$, and so each of $x_8$ and $x_9$ is adjacent to vertices of $H_1$ by exactly 3 blue edges and two red edges.

To this end, one can notice from the above observations that if $x_8$ is adjacent to two nonadjacent vertices of $C_5$ by the red edges, then a red $\theta_5$-graph is produced (Figure 2 depicts the situation), a contradiction. If $x_9$ is adjacent to two nonadjacent vertices of $C_5$ by the red edges, then $x_8$ adjacent to three non-consecutive vertices (of the internal blue cycle) by blue edges. And so a blue $\theta_3$-graph is produced, a contradiction. This completes the proof.

In the following lemma we determine the Ramsey number $R(\theta_4, \theta_5)$. 

![Figure 1: This figure represents the red edges in $K_9$.](image)
Lemma 2.4. The Ramsey number $R(\theta_4, \theta_5) = 9$.

Proof. It is enough to show that $R(\theta_4, \theta_5) \leq 9$. Let a red-blue coloring of $K_9$ be given. By Lemma 2.3, $K_9$ must contain a blue or a red $\theta_5$-graph. If $K_9$ contains a blue $\theta_5$-graph, then we are done. So, suppose $K_9$ has a red $\theta_5$-graph. Let $T_1 = x_1 x_2 x_3 x_1$ be the red triangle. Let $y_1, y_2, \ldots, y_6$ be the remaining vertices. By Theorem 1.1, $H = K_9[y_1, y_2, \ldots, y_6]$ has a red or a blue $C_3$. We consider the following two cases.

Case 1. $H$ contains a blue $C_3$. Let $T_2 = y_1 y_2 y_3 y_1$ be the blue $C_3$. Every vertex in the blue $C_3$ is adjacent at least by two blue edges to $T_1$, as otherwise $K_9$ would have a red $\theta_4$-graph. Let $d_{T_2, \text{blue}}(x_i)$ denote the number of blue edges from $x_i$ to $T_2$. We consider 3 subcases according to the number of blue edges of $x_1, x_2$, and $x_3$ to $T_2$.

Subcase 1.1. $d_{T_2, \text{blue}}(x_1) = d_{T_2, \text{blue}}(x_2) = d_{T_2, \text{blue}}(x_3) = 2$, then a blue $\theta_5$-graph is produced. Figure 3 depicts the situation.

Subcase 1.2. $d_{T_2, \text{blue}}(x_1) = 3 = d_{T_2, \text{blue}}(x_2)$ and $d_{T_2, \text{blue}}(x_3) = 0$, then a blue $\theta_5$-graph is produced. Figure 4 depicts the situation.

Subcase 1.3. $d_{T_2, \text{blue}}(x_1) = 1, d_{T_2, \text{blue}}(x_2) = 3$, and $d_{T_2, \text{blue}}(x_3) = 2$, then a blue $\theta_5$-graph is produced. Figure 5 depicts the situation.

Case 2. $H$ contains a red $C_3$, say $T_2 = y_1 y_2 y_3 y_1$. Let $y_4, y_5, y_6$ be the remaining vertices of $K_9$. Observe that each vertex of $y_4, y_5, y_6$ is adjacent to at least two vertices of each $T_1$ and $T_2$ which are colored by blue, as otherwise, $G$ contains a red $\theta_4$-graph. Hence, if $K_9[y_4, y_5, y_6]$ has a blue edge, say $y_3 y_6$, then a blue $C_3$ is produced. Hence, by the above case a blue $\theta_5$-graph is produced. So, we need to consider the case when $K_9[y_4, y_5, y_6]$ has no blue edges.
Observe that the number of blue edges in $K_9$ is at least 18. Further, the induced graph by blue edges is non-bipartite. By Theorem 1.5, $K_9$ would have a blue $\theta_5$-graph. This completes the proof.

In the following lemma we determine the Ramsey number $R(\theta_4, \theta_6)$.

**Lemma 2.5.** The Ramsey number $R(\theta_4, \theta_6) = 11$.

**Proof.** It is enough to show that $R(\theta_4, \theta_6) \leq 11$. Let a red-blue coloring of $K_{11}$ be given that contains neither a red $\theta_4$ nor a blue $\theta_6$. Suppose $K_{11}$ has a blue cycle of length 6. Let $x_1, x_2, \ldots, x_{11}$ be the vertices of $K_{11}$, and assume $H_1 = x_1x_2x_3x_4x_5x_6x_1$ is the blue $C_6$. Then $H_1$ has no blue chord as otherwise a blue $\theta_6$-graph is produced. So, $K_{11}[x_1, x_3, x_5]$ and $K_{11}[x_2, x_4, x_6]$ are red triangles. Every vertex of the remaining vertices must be adjacent to $H_1$ by at least 4 blue edges as otherwise a red $\theta_4$-graph is produced. Now, let $x_7$ be a vertex of the remaining vertices that is adjacent to $H_1$ by 4 blue edges. We consider three cases.

**Case 1.** $x_7$ is adjacent to 4 consecutive vertices in $H_1$. Assume $x_7$ is adjacent to $x_1, x_2, x_3,$ and $x_4$, then a blue $\theta_6$-graph is produced.

**Case 2.** $x_7$ is adjacent to 3 consecutive vertices and a vertex separated in $H_1$. Assume $x_7$ is adjacent to $x_2, x_3, x_4,$ and $x_6$, then a blue $\theta_6$-graph is produced.
Case 3. $x_7$ is adjacent to a pair of 2 consecutive vertices separated from each other in $H_1$. Assume $x_7$ is adjacent to $x_3, x_4$ and $x_1, x_6$, then a blue $\theta_6$-graph is produced.

So, we need to consider that $K_{11}$ has no blue $C_6$. We need to prove that $K_{11}$ has a red $\theta_4$-graph. By contradiction, suppose $K_{11}$ has no red $\theta_4$-graph. By Theorem 1.1, $K_{11}$ has a red $C_3$. So, the subgraph induced by red edges is a non-bipartite graph. By Theorem 1.4, the number of red edges is at most 27. So, the number of blue edges is at least 28. By Lemma 2.1, $K_{11}$ has a blue $C_3$. Hence, the subgraph induced by blue edges is a non-bipartite graph. By Theorem 1.3, there is a blue $C_6$, this is a contradiction. This completes the proof.

In the following theorem we determine the Ramsey number $R(\theta_4, \theta_k)$, for $k \geq 7$.

**Theorem 2.6.** The Ramsey number $R(\theta_4, \theta_k) = 2k - 1$, $k \geq 7$.

**Proof.** It is enough to show that $R(\theta_4, \theta_k) \leq 2k - 1$, $k \geq 7$. We prove it by contradiction. Let a red-blue coloring of $K_{2k-1}$ be given. Suppose $K_{2k-1}$ has a blue cycle of length $k$. Let $x_1, x_2, \ldots, x_{2k-1}$ be the vertices of $K_{2k-1}$, and assume $H = x_1x_2 \cdots x_{k}x_1$ is the blue $C_k$. Then $H$ has no blue chord as otherwise a blue $\theta_k$-graph is produced. So, $H$ contains a red $\theta_4$-graph. This is a contradiction.

Now, we need to consider the case when $K_{2k-1}$ has no blue cycle of length $k$. By Theorem 1.1, $K_{2k-1}$ contains a red $C_3$. Let $G_1$ be the induced subgraph of the blue edges. Note that the subgraph induced by the red edges is a non-bipartite graph and contains no red $\theta_4$. Hence, the number of red edges is at most $(2k - 2)^2/4 + 2$. Thus, the number of blue edges is

$$\mathcal{E}(G_1) \geq \frac{(2k - 1)(2k - 2)}{2} - \frac{(2k - 2)^2}{4} - 2$$

$$= k^2 - k - 2$$

$$> k^2 - 2k + 3$$

$$\geq \frac{(2k - 2)^2}{4} + 2.$$  \hspace{1cm} (2.1)

Observe that $G_1$ is a non-bipartite graph ($R(\theta_4, C_3) = 7$ and $K_{2k-1}$ does not contain a red $\theta_4$-graph and so it contains a blue $C_3$, Lemma 2.1). If $c(G_1) \geq k$, then by Theorem 1.3, $G_1$ has a blue $C_k$, this is a contradiction. If $c(G_1) \leq k - 1$, then by Theorem 1.2

$$\mathcal{E}(G_1) \leq k^2 - 2k + 1,$$  \hspace{1cm} (2.2)

which contradicts the inequality (2.1) for $k \geq 7$. This completes the proof.

In the following theorem we determine the Ramsey number $R(\theta_5, \theta_k)$, for $k \geq 6$.

**Theorem 2.7.** The Ramsey number $R(\theta_5, \theta_k) = 2k - 1$, $k \geq 6$.

**Proof.** It is enough to show that $R(\theta_5, \theta_k) \leq 2k - 1$, $k \geq 7$. We prove it by contradiction. Let a red-blue coloring of $K_{2k-1}$ be given. Suppose $K_{2k-1}$ has a blue cycle of length $k$. Let $x_1, x_2, \ldots, x_{2k-1}$ be the vertices of $K_{2k-1}$, and assume $H = x_1x_2 \cdots x_{k}x_1$ is the blue $C_k$. Then $H$
has no blue chord as otherwise a blue $\theta_k$-graph is produced. So, $H$ contains a red $\theta_5$-graph. This a contradiction.

Now, we consider the case when $K_{2k-1}$ has no blue $C_k$. Now, we have the following observations.

(i) $R(\theta_4, \theta_k) = 2k - 1, k \geq 7$. Thus, the induced graph on the red edges is a non-bipartite graph.

(ii) $R(\theta_5, \theta_k) = 9$. Thus, the induced graph by the blue edges is a non-bipartite graph.

Let $G_1$ be the graph induced by the blue edges. Since $K_{2k-1}$ has no red $\theta_5$-graph, by Theorem 1.5 the number of red edges is at most $(2k-2)^2/4+1$. Thus, as in the above theorem,

$$\mathcal{E}(G_1) \geq \frac{(2k-2)(2k-1)}{2} - \frac{(2k-2)^2}{4} - 1$$

$$= k^2 - k - 1$$

edges. If $c(G_1) \geq k$, then by Theorem 1.3, there is a blue $C_k$. This is a contradiction. If $c(G_1) \leq k - 1$, then

$$\mathcal{E}(G_1) \leq k^2 - 2k + 1. \quad (2.4)$$

This contradicts the inequality (2.3) for $k \geq 6$. \qed

**Theorem 2.8.** For $n \geq 6$,

$$R(\theta_n, \theta_n) = \begin{cases} 
\left(\frac{3n}{2}\right) - 1 & \text{if } n \text{ is even,} \\
2n - 1 & \text{if } n \text{ is odd.}
\end{cases} \quad (2.5)$$

**Proof.** First we consider the case when $n$ is odd. In $K_{2n-2}$ color the edges of two vertex disjoint complete graphs of order $n - 1$ with a red color and the remaining edges with a blue color. This coloring contains neither a red nor a blue $\theta_n$-graph. We conclude that $R(\theta_n, \theta_n) \geq 2n - 1$.

Let a red-blue coloring of $K_{2n-1}$ be given that contains neither a red nor a blue $\theta_n$. We know, by Theorem 1.1, that $R(C_n, C_n) = 2n - 1$. Thus, $K_{2n-1}$ contains either a red or a blue $C_n$. Without loss of generality, we suppose that $K_{2n-1}$ has a blue $C_n$. Then there are no chords in $C_n$ as otherwise a blue $\theta_n$ is produced. So, $K_{2n-1}$ contains a red $\theta_n$. This is a contradiction.

Now we consider the case when $n$ is even. Let $K_{(3n/2)-2}$ be colored with two colors, say red and blue, as follows: the edges of vertex disjoint $K_{n-1}$ and $K_{n/2-1}$ are colored blue and the remaining edges are colored red. This coloring contains neither a red nor a blue $\theta_n$. We conclude that $R(\theta_n, \theta_n) \geq (3n/2) - 1$.

To show that $R(\theta_n, \theta_n) \leq (3n/2) - 1$, we follow, word by word, the above argument when $n$ is odd by taking into account that $R(C_n, C_n) = (3n/2) - 1$. The proof is complete. \qed

We conclude this paper by highlighting an interesting open problem. We begin with the following constructions.
Let \( n \geq m \geq 6 \). If \( \theta_m \) contains an odd cycle, then in \( K_{2n-2} \) color the edges of a complete bipartite graph \( K_{n-1,n-1} \) with blue and all the remaining edges with red. This coloring contains neither a red \( \theta_n \)-graph nor a blue \( \theta_m \)-graph. Thus, \( R(\theta_n, \theta_m) \geq 2n - 1 \).

Now, we consider the case when \( \theta_m \) contains no odd cycle. If \( n \) is even, then in \( K_{n+\lfloor m/2 \rfloor - 2} \) color the edges of a complete bipartite graph \( K_{n-\lfloor m/2 \rfloor - 1, n-\lfloor m/2 \rfloor - 1} \) with blue and all the remaining edges with red. This coloring contains neither a red \( \theta_n \)-graph nor a blue \( \theta_m \)-graph. Thus, \( R(\theta_n, \theta_m) \geq n + \lfloor m/2 \rfloor - 1 \). If \( n \) is odd, then in \( K_{2m-2} \) color the edges of a complete bipartite graph \( K_{m-1,m-1} \) with red and all the remaining edges with blue and in \( K_{n+\lfloor m/2 \rfloor - 2} \) color the edges of a complete bipartite graph \( K_{n-\lfloor m/2 \rfloor - 1, n-\lfloor m/2 \rfloor - 1} \) with blue and all the remaining edges with red. \( K_{m-1,m-1} \) and \( K_{n-\lfloor m/2 \rfloor - 1, n-\lfloor m/2 \rfloor - 1} \), respectively, provide examples for lower bounds when \( n \) is odd, respectively, when \( \theta_m \) contains no an odd cycle. Thus, \( R(\theta_n, \theta_m) \geq \max\{n + \lfloor m/2 \rfloor - 1, 2m - 1\} \).

From the above construction we conjecture that for \( n \geq m \geq 6 \),

\[
R(\theta_n, \theta_m) = \begin{cases} 
2n - 1 & \text{if } \theta_m \text{ contains odd cycle,} \\
\max\left\{ n + \frac{m}{2} - 1, 2m - 1 \right\} & \text{if } n \text{ is odd and } \theta_m \text{ does not contain an odd cycle,} \\
n + \frac{m}{2} - 1 & \text{if } n \text{ is even and } \theta_m \text{ does not contain an odd cycle.}
\end{cases}
\]

(2.6)

References
