Research Article
Cayley Graphs of Order $27p$ Are Hamiltonian

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Suppose that $G$ is a finite group, such that $|G| = 27p$, where $p$ is prime. We show that if $S$ is any generating set of $G$, then there is a Hamiltonian cycle in the corresponding Cayley graph $Cay(G; S)$.

1. Introduction

Theorem 1.1. If $|G| = 27p$, where $p$ is prime, then every connected Cayley graph on $G$ has a Hamiltonian cycle.

Combining this with results in [1–3] establishes that

\begin{equation}
\text{Every Cayley graph on } G \text{ has a hamiltonian cycle if } |G| = kp, \text{ where } p \text{ is prime, } 1 \leq k < 32, \text{ and } k \neq 24.
\end{equation}

The remainder of the paper provides a proof of the theorem. Here is an outline. Section 2 recalls known results on hamiltonian cycles in Cayley graphs; Section 3 presents the proof under the assumption that the Sylow $p$-subgroup of $G$ is normal; Section 4 presents the proof under the assumption that the Sylow $p$-subgroups of $G$ are not normal.

2. Preliminaries: Known Results on Hamiltonian Cycles in Cayley Graphs

For convenience, we record some known results that provide hamiltonian cycles in various Cayley graphs, after fixing some notation.
Notation 1 (see [4, Sections 1.1 and 5.1]). For any group $G$, we use the following notation:

1. $G'$ denotes the commutator subgroup $[G, G]$ of $G$,
2. $Z(G)$ denotes the center of $G$,
3. $\Phi(G)$ denotes the Frattini subgroup of $G$.

For $a, b \in G$, we use $a^b$ to denote the conjugate $b^{-1}ab$.

Notation 2. If $(s_1, s_2, \ldots, s_n)$ is any sequence, we use $(s_1, s_2, \ldots, s_n)\#$ to denote the sequence $(s_1, s_2, \ldots, s_{n-1})$ that is obtained by deleting the last term.

Theorem 2.1 (Marušič, Durnberger, Keating-Witte [5]). If $G'$ is a cyclic group of prime-power order, then every connected Cayley graph on $G$ has a hamiltonian cycle.

Lemma 2.2 (see [3, Lemma 2.27]). Let $S$ generate the finite group $G$, and let $s \in S$. If

1. $\langle s \rangle \triangleleft G$,
2. Cay($G/\langle s \rangle ; S$) has a hamiltonian cycle, and
3. either
   1. $s \in Z(G)$, or
   2. $|s|$ is prime,

then Cay($G; S$) has a hamiltonian cycle.

Lemma 2.3 (see [1, Lemma 2.7]). Let $S$ generate the finite group $G$, and let $s \in S$. If

1. $\langle s \rangle \triangleleft G$,
2. $|s|$ is a divisor of $pq$, where $p$ and $q$ are distinct primes,
3. $s^p \in Z(G)$,
4. $|G/\langle s \rangle|$ is divisible by $q$, and
5. Cay($G/\langle s \rangle ; S$) has a hamiltonian cycle,

then there is a hamiltonian cycle in Cay($G; S$).

The following results are well known (and easy to prove).

Lemma 2.4 (“Factor Group Lemma”). Suppose that

1. $S$ is a generating set of $G$,
2. $N$ is a cyclic, normal subgroup of $G$,
3. $(s_1N, \ldots, s_nN)$ is a hamiltonian cycle in Cay($G/N; S$), and
4. the product $s_1s_2\cdots s_n$ generates $N$.

Then $(s_1, \ldots, s_n)^{N}$ is a hamiltonian cycle in Cay($G; S$).

Corollary 2.5. Suppose that

1. $S$ is a generating set of $G$,
(ii) $N$ is a normal subgroup of $G$, such that $|N|$ is prime,
(iii) $s \equiv t \pmod{N}$ for some $s, t \in S \cup S^{-1}$ with $s \neq t$, and
(iv) there is a hamiltonian cycle in $\text{Cay}(G/N; S)$ that uses at least one edge labelled $s$.

Then there is a hamiltonian cycle in $\text{Cay}(G; S)$.

**Definition 2.6.** If $H$ is any subgroup of $G$, then $H \setminus \text{Cay}(G; S)$ denotes the multigraph in which

(i) the vertices are the right cosets of $H$, and
(ii) there is an edge joining $Hg_1$ and $Hg_2$ for each $s \in S \cup S^{-1}$, such that $g_1s \in Hg_2$.

Thus, if there are two different elements $s_1$ and $s_2$ of $S \cup S^{-1}$, such that $g_1s_1$ and $g_1s_2$ are both in $Hg_2$, then the vertices $Hg_1$ and $Hg_2$ are joined by a double edge.

**Lemma 2.7** (see [3, Corollary 2.9]). Suppose that

(i) $S$ is a generating set of $G$,
(ii) $H$ is a subgroup of $G$, such that $|H|$ is prime,
(iii) the quotient multigraph $H \setminus \text{Cay}(G; S)$ has a hamiltonian cycle $C$, and
(iv) $C$ uses some double-edge of $H \setminus \text{Cay}(G; S)$.

Then there is a hamiltonian cycle in $\text{Cay}(G; S)$.

**Theorem 2.8** (see [6, Corollary 3.3]). Suppose that

(i) $S$ is a generating set of $G$,
(ii) $N$ is a normal $p$-subgroup of $G$, and
(iii) $st^{-1} \in N$, for all $s, t \in S$.

Then $\text{Cay}(G; S)$ has a hamiltonian cycle.

**Remark 2.9.** In the proof of our main result, we may assume $p \geq 5$, for otherwise either

(i) $|G| = 54$ is of the form $18q$, where $q$ is prime, and so [3, Proposition 9.1] applies, or
(ii) $|G| = 3^4$ is a prime power, and so the main theorem of [7] applies.

### 3. Assume the Sylow $p$-Subgroup of $G$ Is Normal

**Notation 3.** Let

(i) $G$ be a group of order $27p$, where $p$ is prime, and $p \geq 5$ (see Remark 2.9),
(ii) $S$ be a minimal generating set for $G$,
(iii) $P \cong \mathbb{Z}_p$ be a Sylow $p$-subgroup of $G$,
(iv) $w$ be a generator of $P$, and
(v) $Q$ be a Sylow 3-subgroup of $G$. 
Assumption 3.1. In this section, we assume that $P$ is a normal subgroup of $G$.

Therefore $G$ is a semidirect product:

$$G = Q \rtimes P. \quad (3.1)$$

We may assume that $G'$ is not cyclic of prime order (for otherwise Theorem 2.1 applies). This implies that $Q$ is nonabelian and acts nontrivially on $P$; so

$$G' = Q' \times P \text{ is cyclic of order } 3p. \quad (3.2)$$

Notation 4. Since $Q$ is a 3-group and acts nontrivially on $P \cong \mathbb{Z}_p$, we must have $p \equiv 1 \pmod{3}$. Thus, one may choose $r \in \mathbb{Z}$, such that

$$r^3 \equiv 1 \pmod{p}, \text{ but } r \neq 1 \pmod{p}. \quad (3.3)$$

Dividing $r^3 - 1$ by $r - 1$, we see that

$$r^2 + r + 1 \equiv 0 \pmod{p}. \quad (3.4)$$

3.1. A Lemma That Applies to Both of the Possible Sylow 3-Subgroups

There are only 2 nonabelian groups of order 27, and we will consider them as separate cases, but, first, we cover some common ground.

Note

Since $Q$ is a nonabelian group of order 27, and $G = Q \rtimes P \cong Q \rtimes \mathbb{Z}_p$, it is easy to see that

$$Q' = \Phi(Q) = Z(Q) = Z(G) = \Phi(G). \quad (3.5)$$

Lemma 3.2. Assume that

(i) $s \in (S \cup S^{-1}) \cap Q$, such that $s$ does not centralize $P$, and

(ii) $c \in C_Q(P) \setminus \Phi(Q)$.

Then we may assume that $S$ is either $\{s, cs^{-1}w\}$ or $\{s, c^2sw\}$ or $\{s, csw\}$ or $\{s, c^{-1}sw\}$.

Proof. Since $G/P \cong Q$ is a 2-generated group of prime-power order, there must be an element $a$ of $S$, such that $\{s, a\}$ generates $G/P$. We may write

$$a = s^i c^j z^k, \quad \text{with } 0 \leq i \leq 2, \ 1 \leq j \leq 2, \ z \in Z(Q), \text{ and } 0 \leq k < p. \quad (3.6)$$

Note the following.

(i) By replacing $a$ with its inverse if necessary, we may assume $i \in \{0, 1\}$. 
(ii) By applying an automorphism of $G$ that fixes $s$ and maps $c$ to $cz'$, we may assume that $z$ is trivial (since $(cz')^1 = c'z = c'z$).

(iii) By replacing $w$ with $w^k$ if $k \neq 0$, we may assume $k \in \{0,1\}$.

Thus,

$$a = s^i c^j w^k$$

with $i,k \in \{0,1\}$, and $j \in \{1,2\}$.  

(3.7)

Case 1 (Assume $k = 1$). Then $(s,a) = G$, and so $S = \{s,a\}$. This yields the four listed generating sets.

Case 2 (Assume $k = 0$). Then $(s,a) = Q$, and there must be a third element $b$ of $S$, with $b \notin Q$; after replacing $w$ with an appropriate power, we may write $b = tw$ with $t \in Q$. We must have $t \in (s,\Phi(Q))$, for otherwise $(s,b) = G$ (which contradicts the minimality of $S$). Therefore

$$t = s^i z'$$

with $0 \leq i \leq 2$, and $z' \in \Phi(Q) = Z(G)$.  

(3.8)

We may assume the following.

(i) $i' \neq 0$, for otherwise $b = z'w \in S \cap (Z(G) \times P)$; so Lemma 2.3 applies.

(ii) $i' = 1$, by replacing $b$ with its inverse if necessary.

(iii) $z' \neq e$, for otherwise $s$ and $b$ provide a double edge in $\text{Cay}(G/P; S)$; so Corollary 2.5 applies.

Then $s^{-1}b = z'w$ generates $Z(G) \times P$.

Consider the hamiltonian cycles

$$(a^{-1}, s^3)^1, \quad ((a^{-1}, s^3)^3, b), \quad ((a^{-1}, s^3)^3, b^2)$$

(3.9)

in $\text{Cay}(G/\langle z, w \rangle; S)$. Letting $z'' = (a^{-1}s)^3 \in \langle z \rangle$, we see that their endpoints in $G$ are (resp.)

$$z'', \quad z''(s^{-1}b) = z''z'w, \quad z''(s^{-1}b)^3(s^{-1}b) = z''(z')^2 wsw.$$

(3.10)

The final two endpoints both have a nontrivial projection to $P$ (since $s$, being a 3-element, cannot invert $w$), and at least one of these two endpoints also has a nontrivial projection to $Z(G)$. Such an endpoint generates $Z(G) \times P = \langle z, w \rangle$, and so the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\text{Cay}(G, S)$.

\[\square\]

### 3.2. Sylow 3-Subgroup of Exponent 3

**Lemma 3.3.** Assume that $Q$ is of exponent 3; so

$$Q = \langle x, y, z \mid x^3 = y^3 = z^3 = e, \ [x, y] = z, \ [x, z] = [y, z] = e \rangle.$$  

(3.11)
Then one may assume the following:

1. \( w^x = w^r \), but \( y \) and \( z \) centralize \( P \), and
2. either
   
   (a) \( S = \{x, yw\} \), or
   (b) \( S = \{x, xyw\} \).

**Proof.** (1) Since \( Q \) acts nontrivially on \( P \), and \( \text{Aut}(P) \) is cyclic, but \( Q/\Phi(Q) \) is not cyclic, there must be elements \( a \) and \( b \) of \( Q \setminus \Phi(Q) \), such that \( a \) centralizes \( P \), but \( b \) does not. (And \( z \) must centralize \( P \), because it is in \( Q \).

By applying an automorphism of \( Q \), we may assume \( a = y \) and \( b = x \). Furthermore, we may assume \( w^x = w^r \) by replacing \( x \) with its inverse if necessary.

(2) \( S \) must contain an element that does not centralize \( P \); so we may assume \( x \in S \). By applying Lemma 3.2 with \( s = x \) and \( c = y \), we see that we may assume that \( S \) is

\[
\{x, yw\} \text{ or } \{x, y^2w\} \text{ or } \{x, xyw\} \text{ or } \{x, xy^2w\}.
\] (3.12)

But there is an automorphism of \( G \) that fixes \( x \) and \( w \) and sends \( y \) to \( y^2 \); so we need only consider two of these possibilities. \( \square \)

**Proposition 3.4.** Assume, as usual, that \( |G| = 27p \), where \( p \) is prime, and that \( G \) has a normal Sylow \( p \)-subgroup. If the Sylow 3-subgroup \( Q \) is of exponent 3, then \( \text{Cay}(G; S) \) has a hamiltonian cycle.

**Proof.** We write \( \bar{G} \) for the natural homomorphism from \( G \) to \( \bar{G} = G/P \). From Lemma 3.3(2), we see that we need only consider two possibilities for \( S \).

**Case 1** (Assume \( S = \{x, yw\} \)). For \( a = x \) and \( b = yw \), we have the following hamiltonian cycle in \( \text{Cay}(G/P; S) \):

\[
\begin{align*}
& \bar{x} \xrightarrow{a} \bar{x}^2 \xrightarrow{a} \bar{x}y \xrightarrow{a} \bar{x}y^2 \xrightarrow{a} \bar{x}y^2z \xrightarrow{a} \bar{y}z \xrightarrow{a} \bar{y}x^2z \xrightarrow{a} \bar{x}y \xrightarrow{a} \bar{x}y^2 \xrightarrow{a} \bar{x}z \xrightarrow{a} \bar{y}x^2z \xrightarrow{a} \bar{y}x^2 \xrightarrow{a} \bar{y}x \xrightarrow{a} \bar{y} \xrightarrow{a} \bar{x}.
\end{align*}
\] (3.13)

Its endpoint in \( G \) is

\[
a^2ba^{-2}b^2a^2ba^2b^2a^{-1}b^2ab^{-1}a^2b^{-2} = x^2ywx^2(yw)^2x^2ywx^2ywxywxy^{-1}x^{-1}y^2x^2ywxy^2w^{-1}x^2yw^{-2}.
\] (3.14)
Since the walk is a hamiltonian cycle in \( G/P \), we know that this endpoint is in \( P = \langle w \rangle \). So all terms except powers of \( w \) must cancel. Thus, we need only calculate the contribution from each appearance of \( w \) in this expression. To do this, note that if a term \( w^i \) is followed by a net total of \( j \) appearances of \( x \), then the term contributes a factor of \( w^{ir^j} \) to the product. So the endpoint in \( G \) is

\[
\begin{align*}
&w^{13} w^{2r^2} w^{10} w^{r^3} w^{2r^7} w^{5} w^{3} w^{-r^2} w^{-2}.
&\text{(3.15)}
\end{align*}
\]

Since \( r^3 \equiv 1 \pmod{p} \), this simplifies to

\[
\begin{align*}
w^r w^2 w^r w^2 w^2 w^r w^{-r^2} w^{-2} &= w^{r^2 + r^3 + 2r + r^2 + 1 - r^2 - 2} \\
&= w^{r^2 + r + 1} = w^{r^2 + r + 1} w^r = w^9 w^3 = w^3r.
&\text{(3.16)}
\end{align*}
\]

Since \( p \nmid 3r \), this endpoint generates \( P \); so the Factor Group Lemma 2.4 provides a hamiltonian cycle in \( \text{Cay}(G;S) \).

Case 2 (Assume \( S = \{x, xyw\} \)). For \( a = x \) and \( b = xyw \), we have the hamiltonian cycle

\[
\left( \left( a, b^2 \right)^3 \# a \right)^3
\]

in \( \text{Cay}(G/P;S) \). Its endpoint in \( G \) is

\[
\begin{align*}
\left( \left( ab^2 \right)^3 b^{-1} a \right)^3 &= \left( \left( x(xyw)^2 \right)^3 (xyw)^{-1} x \right)^3 = \left( x(x^2y^2w^{r+1}) \right)^3 (w^{-1}y^{-1}x^{-1}) x^3 \\
&= \left( y^2w^{r+1} \right)^3 (w^{-1}y^{-1})^3 = w^{3(r+1)} w^{-1} y^{-1})^3 = (y^{-1}w^{3r+2})^3 \\
&= w^{3(3r+2)}.
&\text{(3.18)}
\end{align*}
\]

Since we are free to choose \( r \) to be either of the two primitive cube roots of 1 in \( \mathbb{Z}_p \), and the equation \( 3r + 2 = 0 \) has only one solution in \( \mathbb{Z}_p \), we may assume that \( r \) has been selected to make the exponent nonzero. Then the Factor Group Lemma 2.4 provides a hamiltonian cycle in \( \text{Cay}(G;S) \).

\[
\square
\]

### 3.3. Sylow 3-Subgroup of Exponent 9

**Lemma 3.5.** Assume that \( Q \) is of exponent 9; so

\[
Q = \left\langle x, y \mid x^9 = y^3 = e, \ [x, y] = x^3 \right\rangle.
\]

\[
\text{(3.19)}
\]
There are two possibilities for \( G \), depending on whether \( C_Q(P) \) contains an element of order 9 or not.

1. Assume that \( C_Q(P) \) does not contain an element of order 9. Then we may assume that \( y \) centralizes \( P \), but \( w^x = w^y \). Furthermore, we may assume that:
   
   (a) \( S = \{ x, yw \} \), or
   
   (b) \( S = \{ x, xyw \} \).

2. Assume that \( C_Q(P) \) contains an element of order 9. Then we may assume \( x \) centralizes \( P \), but \( w^x = w^y \). Furthermore, we may assume that:
   
   (a) \( S = \{ xw, y \} \),
   
   (b) \( S = \{ xyw, y \} \),
   
   (c) \( S = \{ xy, xw \} \), or
   
   (d) \( S = \{ xy, x^2yw \} \).

Proof. (1) Since \( x \) has order 9, we know that it does not centralize \( P \). But \( x^3 \) must centralize \( P \) (since \( x^3 \) is in \( G' \)). Therefore, we may assume \( w^x = x' \) (by replacing \( x \) with its inverse if necessary). Also, since \( Q/C_Q(P) \) must be cyclic (because \( \text{Aut}(P) \) is cyclic), but \( C_Q(P) \) does not contain an element of order 9, we see that \( C_Q(P) \) contains every element of order 3; so \( y \) must be in \( C_Q(P) \).

Since \( S \) must contain an element that does not centralize \( P \), we may assume \( x \in S \). By applying Lemma 3.2 with \( s = x \) and \( c = y \), we see that we may assume that \( S \) is:

\[
\{ x, yw \} \text{ or } \{ x, y^2w \} \text{ or } \{ x, xyw \} \text{ or } \{ x, xy^2w \}.
\]

(3.20)

The second generating set need not be considered, because \( (y^2w)^{-1} = yw^{-1} = yw' \); so it is equivalent to the first. Also, the fourth generating set can be converted into the third, since there is an automorphism of \( G \) that fixes \( y \), but takes \( x \) to \( xyw \) and \( w \) to \( w^{-1} \).

(2) We may assume \( x \in C_Q(P) \); so \( C_Q(P) = \langle x \rangle \).

We know that \( S \) must contain an element \( s \) that does not centralize \( P \), and there are two possibilities: either

(I) \( s \) has order 3, or

(II) \( s \) has order 9.

We consider these two possibilities as separate cases.

Case I (Assume that \( s \) has order 3). We may assume \( s = y \). Letting \( c = x \), we see from Lemma 3.2 that we may assume \( S \) is either

\[
\{ y, xw \} \text{ or } \{ y, x^2w \} \text{ or } \{ y, yxw \} \text{ or } \{ y, yx^2w \}.
\]

(3.21)

The second and fourth generating sets need not be considered, because there is an automorphism of \( G \) that fixes \( y \) and \( w \), but takes \( x \) to \( x^2 \). Also, the third generating set may be replaced with \( \{ y, xyw \} \), since there is an automorphism of \( G \) that fixes \( y \) and \( w \), but takes \( x \) to \( y^{-1}xy \).
Case II (Assume that $s$ has order 9). We may assume $s = xy$. Letting $e = x$, we see from Lemma 3.2 that we may assume that $S$ is either
\[
\{xy, xw\} \text{ or } \{xy, x^2w\} \text{ or } \{xy, xyxw\} \text{ or } \{xy, xyx^2w\}.
\]
(3.22)

The second generating set is equivalent to $\{xy, xw\}$, since the automorphism of $G$ that sends $x$ to $x^4$, $y$ to $x^{-3}y$, and $w$ to $w^{-1}$ maps it to $\{xy, (xw)^{-1}\}$. The third generating set is mapped to $\{xy, x^2yw\}$ by the automorphism that sends $x$ to $x[x, y]$ and $y$ to $[x, y]^{-1}y$. The fourth generating set need not be considered, because $xyx^2w$ is an element of order 3 that does not centralize $P$, which puts it in the previous case. \qed

**Proposition 3.6.** Assume, as usual, that $|G| = 27p$, where $p$ is prime, and that $G$ has a normal Sylow $p$-subgroup. If the Sylow $3$-subgroup $Q$ is of exponent 9, then $\text{Cay}(G; S)$ has a hamiltonian cycle.

**Proof.** We will show that, for an appropriate choice of $a$ and $b$ in $S \cup S^{-1}$, the walk
\[
(a^3, b^{-1}, a, b^{-1}, a^4, b^2, a^{-2}, b, a^2, b, a^3, b, a^{-1}, b^{-1}, a^{-1}, b^{-2})
\]
(3.23)

provides a hamiltonian cycle in $\text{Cay}(G/P; S)$ whose endpoint in $G$ generates $P$ (so the Factor Group Lemma 2.4 applies).

We begin by verifying two situations in which (3.23) is a hamiltonian cycle.

(HC1) If $|\overline{a}| = 9$, $|\overline{b}| = 3$, and $\overline{a^7} = \overline{a^4}$ in $\overline{G} = G/P$, then we have the hamiltonian cycle:
\[
\overline{a} \rightarrow \overline{a} \rightarrow \overline{a^2} \rightarrow \overline{a^3} \rightarrow \overline{a^4} \rightarrow \overline{a^5} \rightarrow \overline{a^6} \rightarrow \overline{a^7}
\]
(3.24)

(HC2) If $|\overline{a}| = 9$, $|\overline{b}| = 9$, $\overline{a^7} = \overline{a^3}$, and $\overline{b^3} = \overline{a^3}$ in $\overline{G} = G/P$, then we have the hamiltonian cycle:
\[
\overline{a} \rightarrow \overline{a} \rightarrow \overline{a^2} \rightarrow \overline{a^3} \rightarrow \overline{a^4} \rightarrow \overline{a^5} \rightarrow \overline{a^6} \rightarrow \overline{a^7}
\]
(3.25)
To calculate the endpoint in \( G \), fix \( r_1, r_2 \in \mathbb{Z}_p \), with

\[
\omega^a = \omega^{r_1}, \quad \omega^b = \omega^{r_2},
\]

and write

\[
a = aw_1, \quad b = bw_2, \text{ where } a, b \in Q, \quad w_1, w_2 \in P. \tag{3.27}
\]

Note that if an occurrence of \( \omega_i \) in the product is followed by a net total of \( j_1 \) appearances of \( a \) and a net total of \( j_2 \) appearances of \( b \), then it contributes a factor of \( \omega_i^{r_1^{j_1}r_2^{j_2}} \) to the product. (A similar occurrence of \( \omega_i^{-1} \) contributes a factor of \( \omega_i^{-r_1^{j_1}r_2^{j_2}} \) to the product.) Furthermore, since \( r_1^3 \equiv r_2^3 \equiv 1 \pmod{p} \), there is no harm in reducing \( j_1 \) and \( j_2 \) modulo 3.

We will apply these considerations only in a few particular situations.

(E1) Assume \( \omega_1 = e \) (so \( a \in Q \) and \( a = a \)). Then the endpoint of the path in \( G \) is

\[
\begin{aligned}
a^3b^{-1}ab^{-1}a^4b^2a^{-2}ba^2ba^{-1}b^{-1}a^{-1}b^{-2} & = a^3(bw_2)^{-1}a(bw_2)^{-1}a^4(bw_2)^2a^{-2}(bw_2)a^2 \\
 & \times (bw_2)a^3bw_2a^{-1}(bw_2)^{-1}a^{-1}(bw_2)^{-2} \\
 & = a^3\left(\omega_2^{-1}b^{-1}\right)a\left(\omega_2^{-1}b^{-1}\right)a^4(bw_2bw_2)a^{-2}(bw_2)a^2 \\
 & \times (bw_2)a^3bw_2a^{-1}\left(\omega_2^{-1}b^{-1}\right)a^{-1}\left(\omega_2^{-1}b^{-1}\omega_2^{-1}b^{-1}\right).
\end{aligned}
\]

By the above considerations, this simplifies to \( w_2^m \), where

\[
m = -1 - r_1^2r_2 + r_1r_2 + r_1 + r_2^2 + r_1r_2 + r_1 - r_1^2 - r_2 - r_2^2 \\
= -r_1^2r_2 - r_2^2 + 2r_1r_2 + 2r_1 - r_2 - 1. \tag{3.29}
\]

Note the following.

(a) If \( r_1 \neq 1 \) and \( r_2 = 1 \), then \( m \) simplifies to \( 6r_1 \), because \( r_1^2 + r_1 + 1 \equiv 0 \pmod{p} \) in this case.

(b) If \( r_1 \neq 1 \) and \( r_2 \neq 1 \), then \( m \) simplifies to \( 3r_1(r_2+1) \), because \( r_1^2 + r_1 + 1 \equiv r_2^2 + r_2 + 1 \equiv 0 \pmod{p} \) in this case.
(E2) Assume \( w_2 = e \) (so \( b \in Q \) and \( b_1 = b \)). Then the endpoint of the path in \( G \) is

\[
a^3b^{-1}ab^{-1}a^4b^2a^{-2}ba^2ba^{-1}b^{-1}a^{-1}b^{-2} = (aw_1)^3b^{-1}(aw_1)b^{-1}(aw_1)^4b(aw_1)^2b(aw_1)^3b(aw_1)^{-1}b^{-1}(aw_1)^{-1}b^{-2} = (aw_1aw_1aw_1)b^{-1}(aw_1aw_1aw_1aw_1)b^2\left(w_1^{-1}a^{-1}w_1^{-1}a^{-1}\right) \times b(aw_1aw_1)b(aw_1aw_1aw_1)b\left(w_1^{-1}a^{-1}\right)b^{-1}\left(w_1^{-1}a^{-1}\right)b^{-2}.
\]

By the above considerations, this simplifies to \( w_1^m \), where

\[
m = r_1^2 + r_1 + 1 + r_1^2r_2 + r_1r_2^2 + r_1r_2 + r_1^2r_2^2 + r_1^2r_2 + r_1r_2 - r_1 - r_1^2r_2
\]

\[
- r_1^2 + r_1^2r_2 + r_1r_2 + 2r_1r_2 + r_1^2r_2 + r_1r_2 - 1 - r_1^2r_2 \equiv 2r_1^2r_2 + 3r_1r_2 + r_2^2 + r_1^2r_2 + r_1r_2 + r_2 - r_1 + 1. \tag{3.31}
\]

Note the following.

(a) If \( r_1 = 1 \) and \( r_2 \neq 1 \), then \( m \) simplifies to \(-3(r_2+2)\), because \( r_2^2 + r_2 + 1 \equiv 0 \) (mod \( p \) ) in this case.

(b) If \( r_1 \neq 1 \) and \( r_2 \neq 1 \), then \( m \) simplifies to \(-r_1r_2 - 2r_1 + r_2 + 2\), because \( r_2^2 + r_1 + 1 \equiv r_2^2 + r_2 + 1 \equiv 0 \) (mod \( p \) ) in this case.

Now we provide a hamiltonian cycle for each of the generating sets listed in Lemma 3.5.

(1a) If \( C_Q(P) \) has exponent 3, and \( S = \{x, yw\} \), we let \( a = x \) and \( b = yw \) in (HC1). In this case, we have \( w_1 = e, r_1 = r, \) and \( r_2 = 1; \) so (E1(a)) tells us that the endpoint in \( G \) is \( w_1^{3r} \).

(1b) If \( C_Q(P) \) has exponent 3, and \( S = \{x, xyw\} \), we let \( a = x \) and \( b = (xyw)^{-1} \) in (HC2).

In this case, we have \( w_1 = e, r_1 = r, \) and \( r_2 = r^{-1} = r^2; \) so (E1(b)) tells us that the endpoint in \( G \) is \( w_2^{m} \), where

\[
m = 3r_1(r_2 + 1) = 3r\left(r^2 + 1\right) = 3\left(r^3 + r\right) \equiv 3(1 + r) = 3(r + 1)(\text{mod } p). \tag{3.32}
\]

(2a) If \( C_Q(P) \) has exponent 9, and \( S = \{xw, y\} \), we let \( a = xw \) and \( b = y \) in (HC1). In this case, we have \( w_2 = e, r_1 = 1, \) and \( r_2 = r; \) so (E2(a)) tells us that the endpoint in \( G \) is \( w_1^{3(r_2+2)} \).

(2b) If \( C_Q(P) \) has exponent 9, and \( S = \{xyw, y\} \), we let \( a = xyw \) and \( b = y \) in (HC1). In this case, we have \( w_2 = e \) and \( r_1 = r_2 = r; \) so (E2(b)) tells us that the endpoint in \( G \) is \( w_2^{m} \), where

\[
m = -r_1r_2 - 2r_1 + r_2 + 2 = -r^2 - 2r + r + 2 = -(r^2 + r + 1) + 3 \equiv 3(\text{mod } p). \tag{3.33}
\]
(2c) If $C_Q(P)$ has exponent 9, and $S = \{xy, xw\}$, we let $a = xw$ and $b = (xy)^{-1}$ in (HC2). In this case, we have $w_2 = e$, $r_1 = 1$, and $r_2 = r^{-1} = r^2$; so (E2(a)) tells us that the endpoint in $G$ is $w_1^m$, where
\[ m = -3(r_2 + 2) = -3\left( r^2 + 2 \right) \equiv -3(-r + 1 + 2) = 3(r - 1) \pmod{p}. \quad (3.34) \]

(2d) If $C_Q(P)$ has exponent 9, and $S = \{xy, x^2yw\}$, we let $a = xy$ and $b = x^2yw$ in (HC2). In this case, we have $w_1 = e$ and $r_1 = r_2 = r$; so (E1(b)) tells us that the endpoint in $G$ is $w_2^m$, where
\[ m = 3r_1(r_2 + 1) = 3r(r + 1) = 3\left( r^2 + r \right) \equiv 3(-1) = -3 \pmod{p}. \quad (3.35) \]

In all cases, there is at most one nonzero value of $r$ (modulo $p$) for which the exponent of $w_i$ is 0. Since we are free to choose $r$ to be either of the two primitive cube roots of 1 in $\mathbb{Z}_p$, we may assume that $r$ has been selected to make the exponent nonzero. Then the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\text{Cay}(G; S)$. \hfill \Box

4. Assume the Sylow $p$-Subgroups of $G$ Are Not Normal

**Lemma 4.1.** Assume that

(i) $|G| = 27p$, where $p$ is an odd prime, and

(ii) the Sylow $p$-subgroups of $G$ are not normal.

Then $p = 13$, and $G = \mathbb{Z}_{13} \rtimes (\mathbb{Z}_3)^3$, where a generator $\omega$ of $\mathbb{Z}_{13}$ acts on $(\mathbb{Z}_3)^3$ via multiplication on the right by the matrix
\[
W = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\quad (4.1)
\]

Furthermore, we may assume that
\[ S \text{ is of the form } \{\omega^i, \omega^j, \omega^k\}, \quad (4.2) \]

where $\omega = (1, 0, 0) \in (\mathbb{Z}_3)^3$, and
\[ (i, j) \in \{(1, 0), (2, 0), (1, 2), (1, 3), (1, 5), (1, 6), (2, 5)\}. \quad (4.3) \]

**Proof.** Let $P$ be a Sylow $p$-subgroup of $G$, and let $Q$ be a Sylow 3-subgroup of $G$. Since no odd prime divides $3 - 1$ or $3^2 - 1$, and 13 is the only odd prime that divides $3^3 - 1$, Sylow’s Theorem [8, Theorem 15.7, page 230] implies that $p = 13$, and that $N_G(P) = P$; so $G$ must have a normal
p-complement \cite[Theorem 7.4.3]{4}; that is, \( G = P \times Q \). Since \( P \) must act nontrivially on \( Q \) (since \( P \) is not normal), we know that it must act nontrivially on \( Q/\Phi(Q) \) \cite[Theorem 5.3.5, page 180]{4}. However, \( P \) cannot act nontrivially on an elementary abelian group of order 3 or \( 3^2 \), because \(|P| = 13\) is not a divisor of \( 3 - 1 \) or \( 3^2 - 1 \). Therefore, we must have \(|Q/\Phi(Q)| = 3^3\); so \( Q \) must be elementary abelian (and the action of \( P \) is irreducible).

Let \( W \) be the matrix representing the action of \( w \) on \((\mathbb{Z}_3)^3\) (with respect to some basis that will be specified later). In the polynomial ring \( \mathbb{Z}_3[X] \), we have the factorization:

\[
\frac{X^{13} - 1}{X - 1} = \left( X^3 - X - 1 \right) \cdot \left( X^3 + X^2 - 1 \right) \cdot \left( X^3 + X^2 + X - 1 \right) \cdot \left( X^3 - X^2 - X - 1 \right).
\]  

(4.4)

Since \( w^{13} = e \), the minimal polynomial of \( W \) must be one of the factors on the right-hand side. By replacing \( w \) with an appropriate power, we may assume that it is the first factor. Then, choosing any nonzero \( v \in (\mathbb{Z}_3)^3 \), the matrix representation of \( w \) with respect to the basis \( \{v, v^w, v^{w^2}\} \) is \( W \) (the Rational Canonical Form).

Now, let \( \zeta \) be a primitive 13th root of unity in the finite field \( \text{GF}(27) \). Then any Galois automorphism of \( \text{GF}(27) \) over \( \text{GF}(3) \) must raise \( \zeta \) to a power. Since the subgroup of order 3 in \( \mathbb{Z}_3^* \) is generated by the number 3, we conclude that the orbit of \( \zeta \) under the Galois group is \( \{\zeta, \zeta^3, \zeta^9\} \). These must be the 3 roots of one of the irreducible factors on the right-hand side of (4.4). Thus, for any \( k \in \mathbb{Z}_{13}^* \), the matrices \( W^k, W^{3k}, \) and \( W^{13} \) all have the same minimal polynomial; so they are conjugate under \( \text{GL}_3(3) \). That is,

\[
W, W^3, W^9
\]

powers of \( W \) in the same row of the following array are conjugate under \( \text{GL}_3(3) \):

\[
W^2, W^5, W^6
\]

\[
W^4, W^{12}, W^{10}
\]

\[
W^7, W^8, W^{11}.
\]

There is an element \( a \) of \( S \) that generates \( G/Q \cong P \). Then \( a \) has order \( p \); so, replacing it by a conjugate, we may assume \( a \in P = \langle w \rangle \), and so \( a = w^i \) for some \( i \in \mathbb{Z}_{13}^* \). From (4.5), we see that we may assume \( i \in \{1, 2\} \) (perhaps after replacing \( a \) by its inverse).

Now let \( b \) be the second element of \( S \); so we may assume \( b = w^i \) for some \( i \). We may assume \( 0 \leq j \leq 6 \) (by replacing \( b \) with its inverse, if necessary). We may also assume \( j \neq i \), for otherwise \( S \subset aQ \), and so Theorem 2.8 applies.

If \( j = 0 \), then \( (i, j) \) is either \((1, 0)\) or \((2, 0)\), both of which appear in the list; henceforth, let us assume \( j \neq 0 \).

Case 1 (Assume \( i = 1 \)). Since \( j \neq i \), we must have \( j \in \{2, 3, 4, 5, 6\} \).

Note that since \( W^3 \) is conjugate to \( W \) under \( \text{GL}_3(3) \) (since they are in the same row of (4.5)), we know that the pair \( (w, w^4) \) is isomorphic to the pair \( (w^3, (w^3)^4) = (w^3, w^-1) \). By replacing \( b \) with its inverse, and then interchanging \( a \) and \( b \), this is transformed to \( (w, w^3) \). So we may assume \( j \neq 4 \).

Case 2 (Assume \( i = 2 \)). We may assume that \( W^j \) is in the second or fourth row of the table (for otherwise we could interchange \( a \) with \( b \) to enter the previous case. So \( j \in \{2, 5, 6\} \). Since
Proof. From Lemma 4.1, for any $i \neq j$, this implies $j \in \{5, 6\}$. However, since $W^5$ is conjugate to $W^2$ (since they are in the same row of (4.5)), and we have $(w^2)^3 = w^6$ and $(w^5)^3 = w^2$, we see that the pair $(w^2, w^6)$ is isomorphic to $(w^2, w^5)$. So we may assume $j \neq 6$.

Proposition 4.2. If $|G| = 27p$, where $p$ is prime, and the Sylow $p$-subgroups of $G$ are not normal, then $\text{Cay}(G; S)$ has a hamiltonian cycle.

Proof. From Lemma 4.1 (and Remark 2.9), we may assume $G = \mathbb{Z}_1^3 \times (\mathbb{Z}_3)^3$. For each of the generating sets listed in Lemma 4.1, we provide an explicit hamiltonian cycle in the quotient multigraph $P \setminus \text{Cay}(G; S)$ that uses at least one double edge. So Lemma 2.7 applies.

To save space, we use $i_1i_2i_3$ to denote the vertex $P(i_1, i_2, i_3)$.

(i, j) = (1, 0) \(a = w, \; a^{-1} = w^{12}, \; b = (1, 0, 0), \; \text{and} \; b^{-1} = (-1, 0, 0)\)
Double edge: 222 → 022 with $a^{-1}$ and $b$:

\[
\begin{align*}
000 & \xrightarrow{b^{-1}} 200 \quad \xrightarrow{a} 020 \quad \xrightarrow{a} 002 \quad \xrightarrow{a} 220 \quad \xrightarrow{b^{-1}} 120 \quad \xrightarrow{a} 012 \\
221 & \xrightarrow{a} 102 \quad \xrightarrow{b} 202 \quad \xrightarrow{a} 210 \quad \xrightarrow{a} 021 \quad \xrightarrow{a} 112 \quad \xrightarrow{a} 201 \\
101 & \xrightarrow{a^{-1}} 211 \quad \xrightarrow{a^{-1}} 212 \quad \xrightarrow{a^{-1}} 222 \quad \xrightarrow{b} 022 \quad \xrightarrow{b} 122 \quad \xrightarrow{a^{-1}} 121 \\
111 & \xrightarrow{a^{-1}} 011 \quad \xrightarrow{a^{-1}} 110 \quad \xrightarrow{a^{-1}} 001 \quad \xrightarrow{a^{-1}} 010 \quad \xrightarrow{a^{-1}} 100 \quad \xrightarrow{b^{-1}} 000.
\end{align*}
\] (4.6)

(i, j) = (2, 0) \(a = w^2, \; a^{-1} = w^{11}, \; b = (1, 0, 0), \; \text{and} \; b^{-1} = (-1, 0, 0)\)
Double edge: 020 → 220 with $a$ and $b^{-1}$:

\[
\begin{align*}
000 & \xrightarrow{b^{-1}} 200 \quad \xrightarrow{a} 002 \quad \xrightarrow{a} 022 \quad \xrightarrow{a} 212 \quad \xrightarrow{b^{-1}} 112 \quad \xrightarrow{a^{-1}} 210 \\
122 & \xrightarrow{a^{-1}} 111 \quad \xrightarrow{a^{-1}} 110 \quad \xrightarrow{b^{-1}} 010 \quad \xrightarrow{a^{-1}} 201 \quad \xrightarrow{b^{-1}} 101 \quad \xrightarrow{a} 012 \\
102 & \xrightarrow{a} 020 \quad \xrightarrow{b^{-1}} 220 \quad \xrightarrow{a} 222 \quad \xrightarrow{a} 211 \quad \xrightarrow{a} 120 \quad \xrightarrow{a} 221 \\
021 & \xrightarrow{a^{-1}} 202 \quad \xrightarrow{a^{-1}} 121 \quad \xrightarrow{a^{-1}} 011 \quad \xrightarrow{a^{-1}} 001 \quad \xrightarrow{a^{-1}} 100 \quad \xrightarrow{b^{-1}} 000.
\end{align*}
\] (4.7)

(i, j) = (1, 2) \(a = w, \; a^{-1} = w^{12}, \; b = w^2(1, 0, 0), \; \text{and} \; b^{-1} = w^{11}(-1, -1, 1)\)
Double edge: 220 → 022 with $a$ and $b$:

\[
\begin{align*}
000 & \xrightarrow{b^{-1}} 221 \quad \xrightarrow{a^{-1}} 012 \quad \xrightarrow{a^{-1}} 120 \quad \xrightarrow{b^{-1}} 102 \quad \xrightarrow{b^{-1}} 200 \quad \xrightarrow{a} 020 \\
002 & \xrightarrow{a} 220 \quad \xrightarrow{b} 022 \quad \xrightarrow{a} 222 \quad \xrightarrow{b} 011 \quad \xrightarrow{a} 111 \quad \xrightarrow{a} 121 \\
122 & \xrightarrow{a} 202 \quad \xrightarrow{a} 210 \quad \xrightarrow{a} 021 \quad \xrightarrow{a} 112 \quad \xrightarrow{b^{-1}} 101 \quad \xrightarrow{a^{-1}} 211 \\
212 & \xrightarrow{b} 201 \quad \xrightarrow{b} 110 \quad \xrightarrow{a^{-1}} 001 \quad \xrightarrow{a^{-1}} 010 \quad \xrightarrow{a^{-1}} 100 \quad \xrightarrow{b^{-1}} 000.
\end{align*}
\] (4.8)
\[(i, j) = (1, 3) \ a = w, \ a^{-1} = w^{12}, \ b = w^5(1, 0, 0), \text{ and } b^{-1} = w^10(0, 1, -1)\]
Double edge: 200 → 020 with \(a\) and \(b\):
\[
\begin{align*}
000 & \rightarrow^{b^{-1}} 012 \rightarrow^{a^{-1}} 120 \rightarrow^{b^{-1}} 221 \rightarrow^{a} 102 \rightarrow^{a} 200 \rightarrow^{b} 020 \\
\rightarrow^{a} 002 & \rightarrow^{a} 220 \rightarrow^{a} 022 \rightarrow^{a} 222 \rightarrow^{a} 212 \rightarrow^{a} 211 \rightarrow^{a} 101 \\
\rightarrow^{b^{-1}} 201 & \rightarrow^{a^{-1}} 112 \rightarrow^{a^{-1}} 021 \rightarrow^{a^{-1}} 210 \rightarrow^{a^{-1}} 202 \rightarrow^{a^{-1}} 122 \rightarrow^{b} 121 \\
\rightarrow^{a^{-1}} 111 & \rightarrow^{a^{-1}} 011 \rightarrow^{a^{-1}} 110 \rightarrow^{a^{-1}} 001 \rightarrow^{a^{-1}} 010 \rightarrow^{a^{-1}} 100 \rightarrow^{b^{-1}} 000.
\end{align*}
\] (4.9)

\[(i, j) = (1, 5) \ a = w, \ a^{-1} = w^{12}, \ b = w^5(1, 0, 0), \text{ and } b^{-1} = w^8(1, 0, 1)\]
Double edge: 220 → 022 with \(a\) and \(b^{-1}\):
\[
\begin{align*}
000 & \rightarrow^{b^{-1}} 101 \rightarrow^{a} 120 \rightarrow^{a} 012 \rightarrow^{a} 221 \rightarrow^{b^{-1}} 010 \rightarrow^{a} 001 \\
\rightarrow^{a} 110 & \rightarrow^{a} 011 \rightarrow^{a} 111 \rightarrow^{b} 121 \rightarrow^{a} 122 \rightarrow^{b^{-1}} 102 \rightarrow^{a} 200 \\
\rightarrow^{a} 020 & \rightarrow^{a} 002 \rightarrow^{a} 220 \rightarrow^{b^{-1}} 022 \rightarrow^{a} 222 \rightarrow^{a} 212 \rightarrow^{a} 211 \\
\rightarrow^{b} 202 & \rightarrow^{a} 210 \rightarrow^{a} 021 \rightarrow^{a} 112 \rightarrow^{a} 201 \rightarrow^{a} 201 \rightarrow^{a} 100 \rightarrow^{b^{-1}} 000.
\end{align*}
\] (4.10)

\[(i, j) = (1, 6) \ a = w, \ a^{-1} = w^{12}, \ b = w^6(1, 0, 0), \text{ and } b^{-1} = w^7(-1, 1, 1)\]
Double edge: 021 → 210 with \(a^{-1}\) and \(b\):
\[
\begin{align*}
000 & \rightarrow^{b^{-1}} 211 \rightarrow^{b^{-1}} 201 \rightarrow^{a^{-1}} 112 \rightarrow^{a^{-1}} 021 \rightarrow^{b} 210 \rightarrow^{b} 101 \\
\rightarrow^{b} 120 & \rightarrow^{a} 012 \rightarrow^{a} 221 \rightarrow^{a} 102 \rightarrow^{a} 200 \rightarrow^{a} 020 \rightarrow^{a} 002 \\
\rightarrow^{a} 220 & \rightarrow^{a} 022 \rightarrow^{a} 222 \rightarrow^{a} 212 \rightarrow^{b} 202 \rightarrow^{a^{-1}} 122 \rightarrow^{a^{-1}} 121 \\
\rightarrow^{a^{-1}} 111 & \rightarrow^{a^{-1}} 011 \rightarrow^{a^{-1}} 110 \rightarrow^{a^{-1}} 001 \rightarrow^{a^{-1}} 010 \rightarrow^{a^{-1}} 100 \rightarrow^{b^{-1}} 000.
\end{align*}
\] (4.11)

\[(i, j) = (2, 5) \ a = w^2, \ a^{-1} = w^{11}, \ b = w^5(1, 0, 0), \text{ and } b^{-1} = w^6(1, 0, 1)\]
Double edge: 112 → 210 with \(a^{-1}\) and \(b\):
\[
\begin{align*}
000 & \rightarrow^{b^{-1}} 101 \rightarrow^{a} 012 \rightarrow^{b} 102 \rightarrow^{a} 020 \rightarrow^{a} 220 \rightarrow^{a} 222 \\
\rightarrow^{b} 112 & \rightarrow^{b} 210 \rightarrow^{a^{-1}} 122 \rightarrow^{a^{-1}} 111 \rightarrow^{a^{-1}} 110 \rightarrow^{a^{-1}} 010 \rightarrow^{a^{-1}} 201 \\
\rightarrow^{a^{-1}} 021 & \rightarrow^{a^{-1}} 202 \rightarrow^{b^{-1}} 211 \rightarrow^{a} 120 \rightarrow^{a} 221 \rightarrow^{a} 200 \rightarrow^{a} 002 \\
\rightarrow^{a} 022 & \rightarrow^{a} 212 \rightarrow^{b^{-1}} 121 \rightarrow^{a^{-1}} 011 \rightarrow^{a^{-1}} 001 \rightarrow^{a^{-1}} 100 \rightarrow^{b^{-1}} 000.
\end{align*}
\] (4.12)

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References

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