Research Article

Q-Functions on Quasimetric Spaces and Fixed Points for Multivalued Maps

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Received 14 December 2010; Revised 26 January 2011; Accepted 31 January 2011

Academic Editor: Qamrul Hasan Ansari

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We discuss several properties of $Q$-functions in the sense of Al-Homidan et al. In particular, we prove that the partial metric induced by any $T_0$ weighted quasipseudometric space is a $Q$-function and show that both the Sorgenfrey line and the Kofner plane provide significant examples of quasimetric spaces for which the associated supremum metric is a $Q$-function. In this context we also obtain some fixed point results for multivalued maps by using Bianchini-Grandolfi gauge functions.

1. Introduction and Preliminaries

Kada et al. introduced in [1] the concept of $w$-distance on a metric space and extended the Caristi-Kirk fixed point theorem [2], the Ekeland variation principle [3] and the nonconvex minimization theorem [4], for $w$-distances. Recently, Al-Homidan et al. introduced in [5] the notion of $Q$-function on a quasimetric space and then successfully obtained a Caristi-Kirk-type fixed point theorem, a Takahashi minimization theorem, an equilibrium version of Ekeland-type variational principle, and a version of Nadler’s fixed point theorem for a $Q$-function on a complete quasimetric space, generalizing in this way, among others, the main results of [1] because every $w$-distance is, in fact, a $Q$-function. This interesting approach has been continued by Hussain et al. [6], and by Latif and Al-Mezel [7], respectively. In particular, the authors of [7] have obtained a nice Rakotch-type theorem for $Q$-functions on complete quasimetric spaces.

In Section 2 of this paper, we generalize the basic theory of $Q$-functions to $T_0$ quasipseudometric spaces. Our approach is motivated, in part, by the fact that in many applications to Domain Theory, Complexity Analysis, Computer Science and Asymmetric Functional Analysis, $T_0$ quasipseudometric spaces (in particular, weightable $T_0$
Our result generalizes and improves, in several ways, well-known fixed point theorems. In quasipseudometric spaces, by using Bianchini-Grandolfi gauge functions in the sense of 

\[\text{Fixed Point Theory and Applications}\]

spaces, play a crucial role in our context we will use the following notion. Throughout this paper the letter \(\mathbb{N}\) and \(\omega\) will denote the set of positive integer numbers and the set of nonnegative integer numbers, respectively.

Our basic references for quasimetric spaces are \([25,26]\). Next we recall several pertinent concepts. By a \(T_0\) quasipseudometric on a set \(X\), we mean a function \(d : X \times X \to [0, \infty)\) such that for all \(x, y, z \in X\),

(i) \(d(x, y) = d(y, x) = 0 \iff x = y\),

(ii) \(d(x, z) \leq d(x, y) + d(y, z)\).

A \(T_0\) quasipseudometric \(d\) on \(X\) that satisfies the stronger condition

(i') \(d(x, y) = 0 \iff x = y\)

is called a quasimetric on \(X\).

We remark that in the last years several authors used the term “quasimetric” to refer to a \(T_0\) quasipseudometric and the term “\(T_1\) quasimetric” to refer to a quasimetric in the above sense.

In the following we will simply write \(T_0\) qpm instead of \(T_0\) quasipseudometric if no confusion arises.

A \(T_0\) qpm space is a pair \((X, d)\) such that \(X\) is a set and \(d\) is a \(T_0\) qpm on \(X\). If \(d\) is a quasimetric on \(X\), the pair \((X, d)\) is then called a quasimetric space.

Given a \(T_0\) qpm \(d\) on a set \(X\), the function \(d^{-1}\) defined by \(d^{-1}(x, y) = d(y, x)\), is also a \(T_0\) qpm on \(X\), called the conjugate of \(d\), and the function \(d^\circ\) defined by \(d^\circ(x, y) = \max\{d(x, y), d^{-1}(x, y)\}\) is a metric on \(X\), called the supremum metric associated to \(d\).

Thus, every \(T_0\) qpm \(d\) on \(X\) induces, in a natural way, three topologies denoted by \(\tau_d\), \(\tau_{d^{-1}}\), and \(\tau_{d^\circ}\), respectively, and defined as follows.

(i) \(\tau_d\) is the \(T_0\) topology on \(X\) which has as a base the family of \(\tau_d\)-open balls \(\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}\), where \(B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}\), for all \(x \in X\) and \(\varepsilon > 0\).

(ii) \(\tau_{d^{-1}}\) is the \(T_0\) topology on \(X\) which has as a base the family of \(\tau_{d^{-1}}\)-open balls \(\{B_{d^{-1}}(x, \varepsilon) : x \in X, \varepsilon > 0\}\), where \(B_{d^{-1}}(x, \varepsilon) = \{y \in X : d^{-1}(x, y) < \varepsilon\}\), for all \(x \in X\) and \(\varepsilon > 0\).

(iii) \(\tau_{d^\circ}\) is the topology on \(X\) induced by the metric \(d^\circ\).

Note that if \(d\) is a quasimetric on \(X\), then \(d^{-1}\) is also a quasimetric, and \(\tau_d\) and \(\tau_{d^{-1}}\) are \(T_1\) topologies on \(X\).

Note also that a sequence \((x_n)_{n \in \mathbb{N}}\) in a \(T_0\) qpm space \((X, d)\) is \(\tau_d\)-convergent (resp., \(\tau_{d^{-1}}\)-convergent) to \(x \in X\) if and only if \(\lim d(x, x_n) = 0\) (resp., \(\lim d(x, x_n) = 0\)).

It is well known (see, for instance, \([26,27]\)) that there exists many different notions of completeness for quasimetric spaces. In our context we will use the following notion.
A $T_0$ qpm space $(X,d)$ is said to be complete if every Cauchy sequence is $\tau_{d,1}$-convergent, where a sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n,x_m) < \varepsilon$ whenever $m \geq n \geq n_\varepsilon$.

In this case, we say that $d$ is a complete $T_0$ qpm on $X$.

2. $Q$-Functions on $T_0$ qpm-Spaces

We start this section by giving the main concept of this paper, which was introduced in [5] for quasimetric spaces.

**Definition 2.1.** A $Q$-function on a $T_0$ qpm space $(X,d)$ is a function $q : X \times X \to [0,\infty)$ satisfying the following conditions:

(Q1) $q(x,z) \leq q(x,y) + q(y,z)$, for all $x,y,z \in X$,

(Q2) if $x \in X, M > 0$, and $(y_n)_{n\in\mathbb{N}}$ is a sequence in $X$ that $\tau_{d,1}$-converges to a point $y \in X$ and satisfies $q(x,y_n) \leq M$, for all $n \in \mathbb{N}$, then $q(x,y) \leq M$,

(Q3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$.

If $(X,d)$ is a metric space and $q : X \times X \to [0,\infty)$ satisfies conditions (Q1) and (Q3) above and the following condition:

(Q2') $q(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous for all $x \in X$, then $q$ is called a $w$-distance on $(X,d)$ (cf. [1]).

Clearly $d$ is a $w$-distance on $(X,d)$ whenever $d$ is a metric on $X$.

However, the situation is very different in the quasimetric case. Indeed, it is obvious that if $(X,d)$ is a $T_0$ qpm space, then $d$ satisfies conditions (Q1) and (Q2), whereas Example 3.2 of [5] shows that there exists a $T_0$ qpm space $(X,d)$ such that $d$ does not satisfy condition (Q3), and hence it is not a $Q$-function on $(X,d)$. In this direction, we next present some positive results.

**Lemma 2.2.** Let $q$ be a $Q$-function on a $T_0$ qpm space $(X,d)$. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d^q(y,z) \leq \varepsilon$.

**Proof.** By condition (Q3), $d(y,z) \leq \varepsilon$. Interchanging $y$ and $z$, it follows that $d(z,y) \leq \varepsilon$, so $d^q(y,z) \leq \varepsilon$. $\square$

**Proposition 2.3.** Let $(X,d)$ be a $T_0$ qpm space. If $d$ is a $Q$-function on $(X,d)$, then $\tau_d = \tau_{d,1}$, and hence, $\tau_d$ is a metrizable topology on $X$.

**Proof.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$ which is $\tau_d$-convergent to some $x \in X$. Then, by Lemma 2.2, $\lim_n d^q(x,x_n) = 0$. We conclude that $\tau_d = \tau_{d,1}$. $\square$

**Remark 2.4.** It follows from Proposition 2.3 that many paradigmatic quasimetrizable topological spaces $(X,\tau)$, as the Sorgenfrey line, the Michael line, the Niemytzki plane and the Kofner plane (see [25]), do not admit any compatible quasimetric $d$ which is a $Q$-function on $(X,d)$.

In the sequel, we show that, nevertheless, it is possible to construct an easy but, in several cases, useful $Q$-function on any quasimetric space, as well as a suitable $Q$-functions on any weightable $T_0$ qpm space.
Recall that the discrete metric on a set $X$ is the metric $d_{01}$ on $X$ defined as $d_{01}(x, x) = 0$, for all $x \in X$, and $d_{01}(x, y) = 1$, for all $x, y \in X$ with $x \neq y$.

**Proposition 2.5.** Let $(X, d)$ be a quasimetric space. Then, the discrete metric on $X$ is a $Q$-function on $(X, d)$.

**Proof.** Since $d_{01}$ is a metric it obviously satisfies condition (Q1) of Definition 2.1.

Now suppose that $(y_n)_{n \in \mathbb{N}}$ is a sequence in $X$ that $\tau_{d^{-1}}$-converges to some $y \in X$, and let $x \in X$ and $M > 0$ such that $d_{01}(x, y_n) \leq M$, for all $n \in \mathbb{N}$. If $M \geq 1$, then $d_{01}(x, y) \leq M$. If $M < 1$, we deduce that $x = y_n$, for all $n \in \mathbb{N}$. Since $\lim_n d(y_n, y) = 0$, it follows that $d(x, y) = 0$, so $x = y$, and thus $d_{01}(x, y) = 0 < M$. Hence, condition (Q2) is also satisfied.

Finally, $d_{01}$ satisfies condition (Q3) taking $\delta = 1/2$ for every $\varepsilon > 0$. \qed

**Example 2.6.** On the set $\mathbb{R}$ of real numbers define $d : \mathbb{R} \times \mathbb{R} \to [0, 1]$ as $d(x, y) = 1$ if $x > y$, and $d(x, y) = \min\{y - x, 1\}$ if $x \leq y$. Then, $d$ is a quasimetric on $\mathbb{R}$ and the topological space $(\mathbb{R}, \tau_d)$ is the celebrated Sorgenfrey line. Since $d^e$ is the discrete metric on $\mathbb{R}$, it follows from Proposition 2.5 that $d^e$ is a $Q$-function on $(\mathbb{R}, d)$.

**Example 2.7.** The quasimetric $d$ on the plane $\mathbb{R}^2$, constructed in Example 7.7 of [25], verifies that $(\mathbb{R}^2, \tau_d)$ is the so-called Kofner plane and that $d^e$ is the discrete metric on $\mathbb{R}^2$, so, by Proposition 2.5, $d^e$ is a $Q$-function on $(\mathbb{R}^2, d)$.

Matthews introduced in [14] the notion of a weightable $T_0$ qpm space (under the name of a “weightable quasimetric space”), and its equivalent partial metric space, as a part of the study of denotational semantics of dataflow networks.

A $T_0$ qpm space $(X, d)$ is called weightable if there exists a function $w : X \to [0, \infty)$ such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. In this case, we say that $d$ is a weightable $T_0$ qpm on $X$. The function $w$ is said to be a weighting function for $(X, d)$ and the triple $(X, d, w)$ is called a weighted $T_0$ qpm space.

A partial metric on a set $X$ is a function $p : X \times X \to [0, \infty)$ such that, for all $x, y, z \in X$:

(i) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,

(ii) $p(x, x) \leq p(x, y)$,

(iii) $p(x, y) = p(y, x)$,

(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair $(X, p)$ such that $X$ is a set and $p$ is a partial metric on $X$.

Each partial metric $p$ on $X$ induces a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon + p(x, x)\}$, for all $x \in X$ and $\varepsilon > 0$.

The precise relationship between partial metric spaces and weightable $T_0$ qpm spaces is provided in the next result.

**Theorem 2.8** (Matthews [14]). (a) Let $(X, d)$ be a weightable $T_0$ qpm space with weighting function. Then, the function $p_d : X \times X \to [0, \infty)$ defined by $p_d(x, y) = d(x, y) + w(x)$, for all $x, y \in X$, is a partial metric on $X$. Furthermore $\tau_d = \tau_{p_d}$.

(b) Conversely, let $(X, p)$ be a partial metric space. Then, the function $d_p : X \times X \to [0, \infty)$ defined by $d_p(x, y) = p(x, y) - p(x, x)$, for all $x, y \in X$ is a weightable $T_0$ qpm on $X$ with weighting function $w$ given by $w(x) = p(x, x)$ for all $x \in X$. Furthermore $\tau_p = \tau_{d_p}$.  \[\hspace{1cm}\]
Remark 2.9. The domain of words, the interval domain, and the complexity quasimetric space provide distinguished examples of theoretical computer science that admit a structure of a weightable $T_0$ qpm space and, thus, of a partial metric space (see, e.g., [14, 20, 21]).

**Proposition 2.10.** Let $(X, d, w)$ be a weighted $T_0$ qpm space. Then, the induced partial metric $p_d$ is a $Q$-function on $(X, d)$.

**Proof.** We will show that $p_d$ satisfies conditions (Q1), (Q2), and (Q3) of Definition 2.1.

(Q1) Let $x, y, z \in X$, then

$$p_d(x, z) \leq p_d(x, y) + p_d(y, z) - p_d(y, y) \leq p_d(x, y) + p_d(y, z).$$ (2.1)

(Q2) Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $X$ which is $\tau_d$-convergent to some $y \in X$. Let $x \in X$ and $M > 0$ such that $p_d(x, y_n) \leq M$, for all $n \in \mathbb{N}$.

Choose $\varepsilon > 0$. Then, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(y_n, y) < \varepsilon$, for all $n \geq n_\varepsilon$. Therefore,

$$p_d(x, y) = d(x, y) + w(x) \leq d(x, y_{n_\varepsilon}) + d(y_{n_\varepsilon}, y) + w(x)$$

$$= p_d(x, y_{n_\varepsilon}) + d(y_{n_\varepsilon}, y) < M + \varepsilon.$$ (2.2)

Since $\varepsilon$ is arbitrary, we conclude that $p_d(x, y) \leq M$.

(Q3) Given $\varepsilon > 0$, put $\delta = \varepsilon/2$. If $p_d(x, y) \leq \delta$ and $p_d(x, z) \leq \delta$, it follows

$$d(y, z) = p_d(y, z) - w(y) \leq p_d(y, z)$$

$$\leq p_d(y, x) + p_d(x, z) \leq 2\delta = \varepsilon. \quad \square$$

**3. Fixed Point Results**

Given a $T_0$ qpm space $(X, d)$, we denote by $2^X$ the collection of all nonempty subsets of $X$, by $Cl_{\tau_d}(X)$ the collection of all nonempty $\tau_d$-closed subsets of $X$, and by $Cl_{\tau_d}(X)$ the collection of all nonempty $\tau_d$-closed subsets of $X$.

Following Al-Homidan et al. [5, Definition 6.1] if $(X, d)$ is a quasimetric space, we say that a multivalued map $T : X \to 2^X$ is $q$-contractive if there exists a $Q$-function $q$ on $(X, d)$ and $r \in [0, 1)$ such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$q(u, v) \leq rq(x, y).$$

Latif and Al-Mezel (see [7]) generalized this notion as follows.

If $(X, d)$ is a quasimetric space, we say that a multivalued map $T : X \to 2^X$ is generalized $q$-contractive if there exists a $Q$-function $q$ on $(X, d)$ such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$q(u, v) \leq k(q(x, y))q(x, y), \quad (3.1)$$

where $k : [0, \infty) \to [0, 1)$ is a function such that $\lim \sup_{t \to 0} k(r) < 1$ for all $t \geq 0$. 
Then, they proved the following improvement of the celebrated Rakotch fixed point theorem (see [28]).

**Theorem 3.1** (Laflit and Al-Mezel [7, Theorem 2.3]). Let \((X,d)\) be a complete quasimetric space. Then, for each generalized \(q\)-contractive multivalued map \(T : X \to Cl_d(X)\) there exists \(z \in X\) such that \(z \in T(z)\).

On the other hand, Bianchini and Grandolfi proved in [29] the following fixed point theorem.

**Theorem 3.2** (Bianchini and Grandolfi [29]). Let \((X,d)\) be a complete metric space and let \(T : X \to X\) be a map such that for each \(x, y \in X\)

\[
d(T(x), T(y)) \leq \varphi(d(x, y)),
\]

where \(\varphi : [0, \infty) \to [0, \infty)\) is a nondecreasing function satisfying \(\sum_{n=0}^{\infty} \varphi^n(t) < \infty\), for all \(t > 0\) (\(\varphi^n\) denotes the \(n\)th iterate of \(\varphi\)). Then, \(T\) has a unique fixed point.

A function \(\varphi : [0, \infty) \to [0, \infty)\) satisfying the conditions of the preceding theorem is called a Bianchini-Grandolfi gauge function (cf [24, 30]).

It is easy to check (see [30, Page 8]) that if \(\varphi\) is a Bianchini-Grandolfi gauge function, then \(\varphi(t) < t\), for all \(t > 0\), and hence \(\varphi(0) = 0\).

Our next result generalizes Bianchini-Grandolfi’s theorem for \(Q\)-functions on complete \(T_0\) qpm spaces.

**Theorem 3.3.** Let \((X,d)\) be a complete \(T_0\) qpm space, \(q\) a \(Q\)-function on \(X\), and \(T : X \to Cl_d(X)\) a multivalued map such that for each \(x, y \in X\) and \(u \in T(x)\), there is \(v \in T(y)\) satisfying

\[
q(u,v) \leq \varphi(q(x,y)),
\]

where \(\varphi : [0, \infty) \to [0, \infty)\) is a Bianchini-Grandolfi gauge function. Then, there exists \(z \in X\) such that \(z \in T(z)\) and \(q(z,z) = 0\).

**Proof.** Fix \(x_0 \in X\) and let \(x_1 \in T(x_0)\). By hypothesis, there exists \(x_2 \in T(x_1)\) such that \(q(x_1,x_2) \leq \varphi(q(x_0,x_1))\). Following this process, we obtain a sequence \((x_n)_{n \in \mathbb{N}}\), with \(x_n \in T(x_{n-1})\) and \(q(x_n,x_{n+1}) \leq \varphi(q(x_{n-1},x_n))\), for all \(n \in \mathbb{N}\). Therefore

\[
q(x_n,x_{n+1}) \leq \varphi^n(q(x_0,x_1)),
\]

for all \(n \in \mathbb{N}\).

Now, choose \(\varepsilon > 0\). Let \(\delta = \delta(\varepsilon) \in (0, \varepsilon)\) for which condition (Q3) is satisfied. We will show that there is \(n_\delta \in \mathbb{N}\) such that \(q(x_n,x_m) < \delta\) whenever \(m > n \geq n_\delta\).

Indeed, if \(q(x_0,x_1) = 0\), then \(\varphi(q(x_0,x_1)) = 0\) and thus \(q(x_n,x_{n+1}) = 0\), for all \(n \in \mathbb{N}\), so, by condition (Q1), \(q(x_n,x_m) = 0\) whenever \(m > n\).
If \( q(x_0, x_1) > 0 \), \( \sum_{n=0}^{\infty} \varphi^n(q(x_0, x_1)) < \infty \), so there is \( n_0 \in \mathbb{N} \) such that

\[
\sum_{n=n_0}^{\infty} \varphi^n(q(x_0, x_1)) < \delta. \tag{3.5}
\]

Then, for \( m > n \geq n_0 \), we have

\[
q(x_n, x_m) \leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\
\leq \varphi^n(q(x_0, x_1)) + \varphi^{n+1}(q(x_0, x_1)) + \cdots + \varphi^{m-1}(q(x_0, x_1)) \\
\leq \sum_{j=n}^{m-1} \varphi^j(q(x_0, x_1)) < \delta. \tag{3.6}
\]

In particular, \( q(x_{n_0}, q_n) \leq \delta \) and \( q(x_{n_0}, q_m) \leq \delta \) whenever \( n, m > n_0 \), so, by Lemma 2.2, \( d^\varphi(x_n, x_m) \leq \varepsilon \) whenever \( n, m > n_0 \).

We have proved that \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( (X, d) \) (in fact, it is a Cauchy sequence in the metric space \( (X, d^\varphi) \)). Since \( (X, d) \) is complete there exists \( z \in X \) such that \( \lim_n d(x_n, z) = 0 \).

Next, we show that \( z \in T(z) \).

To this end, we first prove that \( \lim_n q(x_n, z) = 0 \). Indeed, choose \( \varepsilon > 0 \). Fix \( n \geq n_0 \). Since \( q(x_n, x_m) \leq \delta \) whenever \( m > n \), it follows from condition (Q2) that \( q(x_n, z) \leq \delta < \varepsilon \) whenever \( n \geq n_0 \).

Now for each \( n \in \mathbb{N} \) take \( y_n \in T(z) \) such that

\[
q(x_n, y_n) \leq \varphi(q(x_{n-1}, z)). \tag{3.7}
\]

If \( q(x_{n-1}, z) = 0 \), it follows that \( q(x_n, y_n) = 0 \). Otherwise we obtain \( q(x_n, y_n) < q(x_{n-1}, z) \).

Hence, \( \lim_n q(x_n, y_n) = 0 \), and by Lemma 2.2,

\[
\lim_n d^\varphi(z, y_n) = 0. \tag{3.8}
\]

Therefore, \( z \in \text{Cl}_{d^\varphi}(T(z)) = T(z) \).

It remains to prove that \( q(z, z) = 0 \).

Since \( z \in T(z) \), we can construct a sequence \( (z_n)_{n \in \mathbb{N}} \) in \( X \) such that \( z_1 \in T(z) \), \( z_{n+1} \in T(z_n) \) and

\[
q(z, z_n) \leq \varphi^n(q(z, z)), \quad \forall n \in \mathbb{N}. \tag{3.9}
\]

Since \( \sum_{n=0}^{\infty} \varphi^n(q(z, z)) < \infty \), it follows that \( \lim_n q(z, z_n) = 0 \), and thus \( \lim_n q(z, z_n) = 0 \). So, by Lemma 2.2, \( (z_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( (X, d) \) (in fact, it is a Cauchy sequence in \( (X, d^\varphi) \)). Let \( u \in X \) such that \( \lim_n d(z_n, u) = 0 \). Given \( \varepsilon > 0 \), there is \( n_e \in \mathbb{N} \) such that \( q(z, z_n) \leq \varepsilon \), for all \( n \geq n_e \). By applying condition (Q2), we deduce that \( q(z, u) \leq \varepsilon \), so \( q(z, u) = 0 \). Since \( \lim_n q(x_n, z) = 0 \), it follows from condition (Q1) that \( \lim_n q(x_n, u) = 0 \). Therefore, \( d^\varphi(z, u) \leq \varepsilon \), for all \( \varepsilon > 0 \), by condition (Q3). We conclude that \( z = u \), and thus \( q(z, z) = 0 \).
The next example illustrates Theorem 3.3.

Example 3.4. Let \( X = [0, \pi] \) and let \( d \) be the \( T_0 \) qpm on \( X \) given by \( d(x, y) = \max\{y - x, 0\} \). It is well known that \( d \) is weightable with weighting function \( w \) given by \( w(x) = x \), for all \( x \in X \). Let \( q \) be partial metric induced by \( d \). Then, \( q \) is a \( Q \)-function on \( (X, d) \) by Proposition 2.10.

Note also, that by Theorem 2.8

\[
q(x, y) = \max\{y - x, 0\} + x = \max\{x, y\},
\]

(3.10)

for all \( x, y \in X \). Moreover \((X, d)\) is clearly complete because \( d^n \) is the Euclidean metric on \( X \) and thus \((X, d^n)\) is a compact metric space.

Now define \( T : X \rightarrow Cl_{d^+}(X) \) by

\[
T(x) = \{0\} \cup \left\{ \sin \frac{x}{2n} : n \in \mathbb{N} \right\},
\]

(3.11)

for all \( x \in X \). Note that \( T(x) \notin Cl_{d^+}(X) \) because the nonempty \( \tau_{d^+} \)-closed subsets of \( X \) are the intervals of the form \([0, t], \ t \in X \).

Let \( \varphi : [0, \infty) \rightarrow L_{0, \infty} \) be such that \( \varphi(t) = \sin(t/2) \), for all \( t \in [0, \pi] \), and \( \varphi(t) = t/2 \), for all \( t > \pi \). We wish to show that \( \varphi \) is a Bianchini-Grandolfi gauge function.

It is clear that \( \varphi \) is nondecreasing.

Moreover, \( \sum_{n=0}^{\infty} \varphi^2(t) < \infty \), for all \( t \geq 0 \). Indeed, if \( t > \pi \) we have \( \varphi^2(t) \leq t/2^n \) whenever \( n \in \omega \), while for \( t \in [0, \pi] \), we have \( \varphi(t) \leq t/2 \) so,

\[
\varphi^2(t) = \varphi(\varphi(t)) = \sin \frac{\varphi(t)}{2} \leq \sin \frac{t}{4} \leq \frac{t}{4},
\]

(3.12)

and following this process we deduce the known fact that \( \varphi^2(t) \leq t/2^n \), for all \( n \in \mathbb{N} \). We have shown that \( \varphi \) is a Bianchini-Grandolfi gauge function.

Finally, for each \( x, y \in X \) and \( u \in T(x) \setminus \{0\} \), there exists \( n \in \mathbb{N} \) such that \( u = \sin(x/2n) \).

Choose \( v = \sin(y/2n) \). Then \( v \in T(y) \) and

\[
q(u, v) = \max\left\{ \sin \frac{x}{2n}, \sin \frac{y}{2n} \right\} \leq \max\left\{ \sin \frac{x}{2}, \sin \frac{y}{2} \right\}
\]

(3.13)

\[
= \sin \frac{\max\{x, y\}}{2} = \varphi(\max\{x, y\}) = \varphi(q(x, y)).
\]

If \( u = 0 \), then \( u \in T(y) \), and thus \( q(u, u) = 0 \leq \varphi(q(x, y)) \).

We have checked that conditions of Theorem 3.3 are fulfilled, and hence, there is \( z \in T(z) \) with \( q(z, z) = 0 \). In fact \( z = 0 \) is the only point of \( X \) satisfying \( q(z, z) = 0 \) and \( z \in T(z) \) (actually \( \{z\} = T(z) \)). The following consequence of Theorem 3.3, which is also illustrated by Example 3.4, improves and generalizes in several directions the Banach Contraction Principle for partial metric spaces obtained in Theorem 5.3 of [14].
Remark 3.7. The proof of Theorem 3.3 shows that the condition that

Proof. Since \( p = p_{d_p} \) (see Theorem 2.8), we deduce from Proposition 2.10 that \( p \) is a \( Q \)-function for the complete (weightable) \( T_0 \) qpm space \( (X, d_p) \). The conclusion follows from Theorem 3.3.

Observe that if \( k : [0, \infty) \to [0, 1) \) is a nondecreasing function such that \( \limsup_{t \to \infty} k(t) < 1 \), for all \( t \geq 0 \), then the function \( \varphi : [0, \infty) \to [0, \infty) \) given by \( \varphi(t) = k(t) t \), a Bianchini-Grandolfi gauge function (compare [31, Proposition 8]). Therefore, the following variant of Theorem 3.1, which improves Corollary 2.4 of [7], is now a consequence of Theorem 3.3.

Corollary 3.6. Let \( (X, d) \) be a complete \( T_0 \) qpm space. Then, for each generalized q-contractive multivalued map \( T : X \to \text{Cl}_{d'}(X) \) with \( q \) nondecreasing, there exists \( z \in X \) such that \( z \in T(z) \) and \( q(z, z) = 0 \).

Remark 3.7. The proof of Theorem 3.3 shows that the condition that \( (X, d) \) is complete can be replaced by the more general condition that every Cauchy sequence in the metric space \( (X, d^*) \) is \( \tau_{d^*} \)-convergent.

Acknowledgments

The authors thank one of the reviewers for suggesting the inclusion of a concrete example to which Theorem 3.3 applies. They acknowledge the support of the Spanish Ministry of Science and Innovation, Grant no. MTM2009-12872-C02-01.

References


