Research Article

On T-Stability of Picard Iteration in Cone Metric Spaces

M. Asadi, H. Soleimani, S. M. Vaezpour, and B. E. Rhoades

1 Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran 14778 93855, Iran
2 Department of Mathematics, Amirkabir University of Technology, Tehran 15916 34311, Iran
3 Department of Mathematics, Newcastle University, Newcastle, NSW 2308, Australia
4 Department of Mathematics, Indiana University, Bloomington, IN 46205, USA

Correspondence should be addressed to S. M. Vaezpour, vaez@aut.ac.ir

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The aim of this work is to investigate the T-stability of Picard’s iteration procedures in cone metric spaces and give an application.

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1. Introduction and Preliminary

Let E be a real Banach space. A nonempty convex closed subset $P \subseteq E$ is called a cone in $E$ if it satisfies the following:

(i) $P$ is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}, a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space $E$ can be partially ordered by the cone $P \subseteq E$; by defining, $x \leq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ if $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of $P$.

A cone $P$ is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

In the following we always suppose that $E$ is a real Banach space, $P$ is a cone in $E$, and $\leq$ is the partial ordering with respect to $P$.

Definition 1.1 (see [1]). Let $X$ be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies the following:
Definition 1.2. Let \(d\) be a map for which there exist real numbers \(a, b, c\) satisfying 
\[0 < a < 1, \ 0 < b < 1/2, \ 0 < c < 1/2\]. Then \(T\) is called a Zamfirescu operator if, for each pair 
\(x, y \in X, \ T\) satisfies at least one of the following conditions:

1. \(d(Tx,Ty) \leq ad(x,y)\),
2. \(d(Tx,Ty) \leq b(d(x,Tx) + d(y,Ty))\),
3. \(d(Tx,Ty) \leq c(d(x,Ty) + d(y,Tx))\).

Every Zamfirescu operator \(T\) satisfies the inequality:

\[d(Tx,Ty) \leq \delta d(x,y) + 2\delta d(x,Tx)\]  \((1.1)\) 

for all \(x, y \in X\), where \(\delta = \max\{a,b/(1-b),c/(1-c)\}\), with \(0 < \delta < 1\). For normed spaces see [2].

Lemma 1.3 (see [3]). Let \(\{a_n\}\) and \(\{b_n\}\) be nonnegative real sequences satisfying the following inequality:

\[a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \]  \((1.2)\) 

where \(\lambda_n \in (0,1)\), for all \(n \geq n_0\), \(\sum_{n=1}^{\infty} \lambda_n = \infty\), and \(b_n/\lambda_n \to 0\) as \(n \to \infty\). Then \(\lim_{n \to \infty} a_n = 0\).

Remark 1.4. Let \(\{a_n\}\) and \(\{b_n\}\) be nonnegative real sequences satisfying the following inequality:

\[a_{n+1} \leq \lambda a_{n-m} + b_n, \]  \((1.3)\) 

where \(\lambda \in (0,1)\), for all \(n \geq n_0\) and for some positive integer number \(m\). If \(b_n \to 0\) as \(n \to \infty\). Then \(\lim_{n \to \infty} a_n = 0\).

Lemma 1.5. Let \(P\) be a normal cone with constant \(K\), and let \(\{a_n\}\) and \(\{b_n\}\) be sequences in \(E\) satisfying the following inequality:

\[a_{n+1} \leq ha_n + b_n, \]  \((1.4)\) 

where \(h \in (0,1)\) and \(b_n \to 0\) as \(n \to \infty\). Then \(\lim_{n \to \infty} a_n = 0\).

Proof. Let \(m\) be a positive integer such that \(h^m K < 1\). By recursion we have

\[a_{n+1} \leq b_n + hb_{n-1} + \cdots + h^m b_{n-m} + h^{m+1} a_{n-m}, \]  \((1.5)\)
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so

$$\|a_{n+1}\| \leq K\|b_n + hb_{n-1} + \cdots + h^m b_{n-m}\| + h^{m+1}K\|a_{n-m}\|, \quad (1.6)$$

and then by Remark 1.4 $\|a_n\| \to 0$. Therefore $a_n \to 0$.

\[ \square \]

2. **T-Stability in Cone Metric Spaces**

Let $(X,d)$ be a cone metric space, and $T$ a self-map of $X$. Let $x_0$ be a point of $X$, and assume that $x_{n+1} = f(T,x_n)$ is an iteration procedure, involving $T$, which yields a sequence $\{x_n\}$ of points from $X$.

**Definition 2.1** (see [4]). The iteration procedure $x_{n+1} = f(T,x_n)$ is said to be $T$-stable with respect to $T$ if $\{x_n\}$ converges to a fixed point $q$ of $T$ and whenever $\{y_n\}$ is a sequence in $X$ with $\lim_{n \to \infty} d(y_{n+1}, f(T,y_n)) = 0$ we have $\lim_{n \to \infty} y_n = q$.

In practice, such a sequence $\{y_n\}$ could arise in the following way. Let $x_0$ be a point in $X$. Set $x_{n+1} = f(T,x_n)$. Let $y_0 = x_0$. Now $x_1 = f(T,x_0)$. Because of rounding or discretization in the function $T$, a new value $y_1$ approximately equal to $x_1$ might be obtained instead of the true value of $f(T,x_0)$. Then to approximate $y_2$, the value $f(T,y_1)$ is computed to yield $y_2$, an approximation of $f(T,y_1)$. This computation is continued to obtain $\{y_n\}$ an approximate sequence of $\{x_n\}$.

One of the most popular iteration procedures for approximating a fixed point of $T$ is Picard’s iteration defined by $x_{n+1} = Tx_n$. If the conditions of Definition 2.1 hold for $x_{n+1} = Tx_n$, then we will say that Picard’s iteration is $T$-stable.

Recently Qing and Rhoades [5] established a result for the $T$-stability of Picard’s iteration in metric spaces. Here we are going to generalize their result to cone metric spaces and present an application.

**Theorem 2.2.** Let $(X,d)$ be cone metric space, $P$ a normal cone, and $T : X \to X$ with $F(T) \neq \emptyset$. If there exist numbers $a \geq 0$ and $0 \leq b < 1$, such that

$$d(Tx,q) \leq ad(x,Tx) + bd(x,q) \quad (2.1)$$

for each $x \in X$, $q \in F(T)$ and in addition, whenever $\{y_n\}$ is a sequence with $d(y_{n+1},Ty_{n+1}) \to 0$ as $n \to \infty$, then Picard’s iteration is $T$-stable.

**Proof.** Suppose $\{y_n\} \subseteq X$, $c_n = d(y_{n+1},Ty_{n+1})$ and $c_n \to 0$. We shall show that $y_n \to q$. Since

$$d(y_{n+1},q) \leq d(y_{n+1},Ty_{n+1}) + d(Ty_{n+1},q) \leq c_n + ad(y_n,Ty_n) + bd(y_n,q), \quad (2.2)$$

if we put $a_n := d(Ty_{n+1},q)$ and $b_n := c_n + ad(y_n,Ty_n)$ in Lemma 1.5, then we have $y_n \to q$.

Note that the fixed point $q$ of $T$ is unique. Because $p$ is another fixed point of $T$, then

$$d(p,q) = d(Tp,q) \leq ad(p,Tp) + bd(p,q) = bd(p,q), \quad (2.3)$$

which implies $p = q$.

\[ \square \]
Corollary 2.3. Let \((X,d)\) be a cone metric space, \(P\) a normal cone, and \(T : X \to X\) with \(q \in F(T)\). If there exists a number \(\lambda \in [0,1)\), such that \(d(Tx,Ty) \leq \lambda d(x,y)\), for each \(x, y \in X\), then Picard’s iteration is \(T\)-stable.

Corollary 2.4. Let \((X,d)\) be a cone metric space, \(P\) a normal cone, and \(T : X \to X\) is a Zamfirescu operator with \(F(T) \neq \emptyset\) and whenever \(\{y_n\}\) is a sequence with \(d(y_n, Ty_n) \to 0\) as \(n \to \infty\), then Picard’s iteration is \(T\)-stable.

Definition 2.5 (see [6]). Let \((X,d)\) be a cone metric space. A map \(T : X \to X\) is called a quasicontraction if for some constant \(\lambda \in (0,1)\) and for every \(x, y \in X\), there exists \(u \in C(T;x,y) \equiv \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}\), such that \(d(Tx,Ty) \leq \lambda u\).

Lemma 2.6. If \(T\) is a quasicontraction with \(0 < \lambda < 1/2\), then \(T\) is a Zamfirescu operator and so satisfies (2.1).

Proof. Let \(\lambda \in (0,1/2)\) for every \(x, y \in X\) we have \(d(Tx,Ty) \leq \lambda u\) for some \(u \in C(T;x,y)\). In the case that \(u = d(x,Ty)\) we have

\[
d(Tx,Ty) \leq \lambda d(x,Ty) \leq \lambda d(x,Tx) + \lambda d(Tx,Ty).
\]

So

\[
d(Tx,Ty) \leq \frac{\lambda}{1-\lambda} d(x,Tx) \leq 2 \frac{\lambda}{1-\lambda} d(x,Tx) + \frac{\lambda}{1-\lambda} d(x,y).
\]

Put \(\delta := \lambda/(1-\lambda)\) so \(0 < \delta < 1\). The other cases are similarly proved. Therefore \(T\) is a Zamfirescu operator. \(\square\)

Theorem 2.7. Let \((X,d)\) be a nonempty complete cone metric space, \(P\) be a normal cone, and \(T\) a quasicontraction and self map of \(X\) with some \(0 < \lambda < 1/2\). Then Picard’s iteration is \(T\)-stable.

Proof. By [6, Theorem 2.1], \(T\) has a unique fixed point \(q \in X\). Also \(T\) satisfies (2.1). So by Theorem 2.2 it is enough to show that \(d(y_n, Ty_n) \to 0\). We have

\[
d(y_n, Ty_n) \leq d(y_n, Ty_{n-1}) + d(Ty_{n-1}, Ty_n).
\]

Put \(b_n := d(y_n, Ty_n), c_n := d(y_{n+1}, Ty_n)\) and \(d_n := d(Ty_{n-1}, Ty_n)\). Therefore \(c_n \to 0\) as \(n \to \infty\) and

\[
b_n \leq c_{n-1} + d_n \leq c_{n-1} + \lambda u_n,
\]

where

\[
u_n \in C(T, y_{n-1}, y_n) = \{d(y_{n-1}, y_n), d(y_{n-1}, Ty_{n-1}), d(y_n, Ty_n), d(y_n, Ty_{n-1})\}.
\]
Hence we have \(u_n = b_n\) or \(u_n \leq sb_{n-1} + lc_{n-1}\) where \(s = 0, 1\) or \(1/(1 - \lambda)\) and \(l = 1\) or \(1 + \lambda\). Therefore by (2.7), \(b_n \leq (\lambda l + 1)c_{n-1} + \lambda sb_{n-1}\) by \(0 \leq \lambda s < 1\). Now by Lemma 1.5 we have \(b_n \to 0\).

**3. An Application**

**Theorem 3.1.** Let \(X := (C[0, 1], \mathbb{R})\) with \(\|f\|_\infty := \sup_{0 \leq t \leq 1} |f(x)|\) for \(f \in X\) and let \(T\) be a self map of \(X\) defined by \(Tf(x) = \frac{1}{1} \int_0^t F(x, f(t))\) where

(a) \(F: [0, 1] \times \mathbb{R} \to \mathbb{R}\) is a continuous function,

(b) the partial derivative \(F_y\) of \(F\) with respect to \(y\) exists and \(|F_y(x, y)| \leq L\) for some \(L \in [0, 1]\),

(c) for every real number \(0 \leq a < 1\) one has \(a x \leq F(x, ay)\) for every \(x, y \in [0, 1]\).

Let \(P := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}\) be a normal cone and \((X, d)\) the complete cone metric space defined by \(d(f, g) = (\|f - g\|_\infty, a\|f - g\|_\infty)\) where \(a \geq 0\). Then,

(i) Picard’s iteration is \(T\)-stable if \(0 \leq L < 1/2\),

(ii) Picard’s iteration fails to be \(T\)-stable if \(1/2 \leq L < 1\) and \(\int_0^1 f(x, t)\) does not exist.

**Proof.** (i) We have \(T\) being a continuous quasicontraction map with \(0 \leq \lambda := L < 1/2\); so by Theorem 2.7, Picard’s iteration is \(T\)-stable.

(ii) Put \(y_n(x) := nx/(n + 1)\) so \(y_n \in X\) and \(d(y_n, h) \to 0\), where \(h(x) = x\). Also \(d(y_{n+1}, Ty_n) \to 0\), since

\[
\|y_{n+1} - Ty_n\|_\infty = \sup_{0 \leq t \leq 1} \left| \frac{n + 1}{n + 2} x - \int_0^t F\left(x, \frac{nt}{n + 1}\right) dt \right|
\]

\[
\leq \sup_{0 \leq t \leq 1} \left| \frac{n + 1}{n + 2} x - \frac{nx}{n + 1} \right| \to 0,
\]

as \(n \to \infty\). But \(y_n \to h\) and \(h\) is not a fixed point for \(T\). Therefore Picard’s iteration is not \(T\)-stable.

**Example 3.2.** Let \(F_1(x, y) := x + y/4\) and \(F_2(x, y) := x + y/2\). Therefore \(F_1\) and \(F_2\) satisfy the hypothesis of Theorem 3.1 where \(F_1\) has property (i) and \(F_2\) has property (ii). So the self maps \(T_1, T_2\) of \(X\) defined by \(T_1 f(x) = x + (1/4) \int_0^t f(t) dt\) and \(T_2 f(x) = x + (1/2) \int_0^t f(t) dt\) have unique fixed points but Picard’s iteration is \(T\)-stable for \(T_1\) but not \(T\)-stable for \(T_2\).

**References**


