Research Article

The $C^1$ Solutions of the Series-Like Iterative Equation with Variable Coefficients

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Received 23 March 2009; Revised 11 June 2009; Accepted 6 July 2009

Recommended by Tomas Domínguez Benavides

By constructing a structure operator quite different from that of Zhang and Baker (2000) and using the Schauder fixed point theory, the existence and uniqueness of the $C^1$ solutions of the series-like iterative equations with variable coefficients are discussed.

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1. Introduction

An important form of iterative equations is the polynomial-like iterative equation

$$
\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in I := [a,b],
$$

(1.1)

where $F$ is a given function, $f$ is an unknown function, $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$), and $f^k$ ($k = 1, 2, \ldots, n$) is the $k$th iterate of $f$, that is, $f^0(x) = x$, $f^k(x) = f \circ f^{k-1}(x)$. The case of all constant $\lambda_i$'s was considered in [1–10]. In 2000, W. N. Zhang and J. A. Baker first discussed the continuous solutions of such an iterative equation with variable coefficients $\lambda_i = \lambda_i(x)$ which are all continuous in interval $[a, b]$. In 2001, J. G. Si and X. P. Wang furthermore gave the continuously differentiable solution of such equation in the same conditions as in [11]. In this paper, we continue the works of [11, 12], and consider the series-like iterative equation with variable coefficients

$$
\sum_{i=1}^{\infty} \lambda_i(x) f^i(x) = F(x), \quad x \in I := [a,b],
$$

(1.2)
where \( \lambda_i(x) : I \to [0, 1] \) are given continuous functions and \( \sum_{i=1}^{\infty} \lambda_i(x) = 1, \ \lambda_i(x) \geq c > 0 \ (\forall x \in I) \), \( \max_{x \in I} \lambda_i(x) = c_i \). We improve the methods given by the authors in [11, 12], and the conditions of [11, 12] are weakened by constructing a new structure operator.

### 2. Preliminaries

Let \( C^0(I, R) = \{ f : I \to R, \ f \text{ is continuous} \} \), clearly \( (C^0(I, R), \| \cdot \|_{c_0}) \) is a Banach space, where \( \| f \|_{c_0} = \max_{x \in I} |f(x)| \), for \( f \) in \( C^0(I, R) \).

Let \( C^1(I, R) = \{ f : I \to R, \ f \text{ is continuous and continuously differentiable} \} \), then \( C^1(I, R) \) is a Banach space with the norm \( \| \cdot \|_{c_1} \), where \( \| f \|_{c_1} = \| f \|_{c_0} + \| f' \|_{c_0} \), for \( f \) in \( C^1(I, R) \).

Being a closed subset, \( C^1(I, I) \) defined by

\[
C^1(I, I) = \left\{ f \in C^1(I, R), \ f(I) \subseteq I, \ \forall x \in I \right\}
\]  

(2.1)

is a complete space.

The following lemmas are useful, and the methods of proof are similar to those of paper [4], but the conditions are weaker than those of [4].

**Lemma 2.1.** Suppose that \( \varphi \in C^1(I, I) \) and

\[
\begin{align*}
|\varphi'(x)| & \leq M, \ \forall x \in I, \\
|\varphi'(x_1) - \varphi'(x_2)| & \leq M'|x_1 - x_2|, \ \forall x_1, x_2 \in I,
\end{align*}
\]

(2.2)

(2.3)

where \( M \) and \( M' \) are positive constants. Then

\[
\left| (\varphi^n(x_1))' - (\varphi^n(x_2))' \right| \leq M' \left( \sum_{i=1}^{2n-2} M'^i \right) |x_1 - x_2|,
\]

(2.4)

for any \( x_1, x_2 \) in \( I \), where \( (\varphi^n)' \) denotes \( d\varphi^n/dx \).

**Lemma 2.2.** Suppose that \( \varphi_1, \varphi_2 \in C^1(I, I) \) satisfy (2.2). Then

\[
\| \varphi_1^n - \varphi_2^n \|_{c_0} \leq \left( \sum_{i=1}^{n} M^{i-1} \right) \| \varphi_1 - \varphi_2 \|_{c_0}.
\]

(2.5)

**Lemma 2.3.** Suppose that \( \varphi_1, \varphi_2 \in C^1(I, I) \) satisfy (2.2) and (2.3). Then

\[
\left\| (\varphi_1^{k+1})' - (\varphi_2^{k+1})' \right\|_{c_0} \leq (k + 1)M^k \| \varphi_1' - \varphi_2' \|_{c_0}
\]

\[
+ Q(k + 1)M' \left( \sum_{i=1}^{k} (k - i + 1)M^{k+i-1} \right) \| \varphi_1 - \varphi_2 \|_{c_0},
\]

(2.6)

for \( k = 0, 1, 2, \ldots \), where \( Q(s) = 0 \) as \( s = 1 \) and \( Q(s) = 1 \) as \( s = 2, 3, \ldots \).
3. Main Results

For given constants $M_1 > 0$ and $M_2 > 0$, let

$$\mathcal{A}(M_1, M_2) = \left\{ \varphi \in C^1(I, I) : |\varphi'(x)| \leq M_1, \forall x \in I, \right.$$ \hspace{1cm} (3.1)

$$\left. |\varphi'(x_1) - \varphi'(x_2)| \leq M_2|x_1 - x_2|, \forall x_1, x_2 \in I \right\}.$$

**Theorem 3.1** (existence). Given positive constants $M_1$, $M_2$ and $F \in \mathcal{A}(M_1, M_2)$, if there exists constants $N_1 \geq 1$ and $N_2 > 0$, such that

(P1) $c - \sum_{i=1}^{\infty} \lambda_i N_i^1 \geq M_1/N_1$,

(P2) $c - \sum_{i=2}^{\infty} \lambda_i (\sum_{j=1}^{2i-2} N_j^i) \geq M_2/N_2$,

then (1.2) has a solution $f$ in $\mathcal{A}(N_1, N_2)$.

**Proof.** For convenience, let $d = \max\{|a|, |b|\}$.

Define $K : \mathcal{A}(N_1, N_2) \rightarrow C^1(I, I)$ such that $K : f \rightarrow K_f$, where

$$K_f(t) = \sum_{i=1}^{\infty} \lambda_i(x) f^i(t), \quad \forall x, t \in I.$$ \hspace{1cm} (3.2)

Since $f \in \mathcal{A}(N_1, N_2)$, it is easy to see that $|f^i(t)| \leq d$ for all $t \in I$, and $|\lambda_i(x) f^i(t)| \leq d|\lambda_i(x)|$ for all $x, t \in I$. It follows from $\sum_{i=1}^{\infty} \lambda_i(x) = 1$ that $\sum_{i=1}^{\infty} \lambda_i(x) f^i(t)$ is uniformly convergent. Then $K_f(t)$ is continuous for $t \in I$. Also we have

$$a = \sum_{i=1}^{\infty} \lambda_i(x) a \leq \sum_{i=1}^{\infty} \lambda_i(x) f^i(t) \leq \sum_{i=1}^{\infty} \lambda_i(x) b = b,$$ \hspace{1cm} (3.3)

thus $K_f \in C^0(I, I)$.

For any $f \in \mathcal{A}(N_1, N_2)$, we have

$$\left| \frac{d}{dt} \left( \lambda_i(x) \left( f^i(t) \right) \right) \right| = \lambda_i(x) \left| f^i(t) \right| \leq c_i N_i^1.$$ \hspace{1cm} (3.4)

By condition (P1), we see that $\sum_{i=1}^{\infty} c_i N_i^1$ is convergent, therefore $\sum_{i=1}^{\infty} c_i (f^i(t))'$ is uniformly convergent for $t \in I$, this implies that $K_f(t)$ is continuously differentiable for $t \in I$. Moreover

$$\left| \frac{d}{dt} K_f(t) \right| \leq \sum_{i=1}^{\infty} \lambda_i(x) \left| (f^i(t))' \right| \leq \sum_{i=1}^{\infty} c_i N_i^1 := \mu_i.$$ \hspace{1cm} (3.5)
By Lemma 2.1,

\[
\left| \frac{d}{dt} (K_f(t_1)) - \frac{d}{dt} (K_f(t_2)) \right| \leq \sum_{i=1}^{\infty} \lambda_i(x) \left| (f'(t_1))' - (f'(t_2))' \right|
\]

\[
\leq \sum_{i=1}^{\infty} c_i \left( N_2 \sum_{j=1}^{2i-2} N_1^j \right) |t_1 - t_2| := \mu_2 |t_1 - t_2|.
\]

Thus \( K_f \in \mathcal{A}(\mu_1, \mu_2) \).

Define \( T : \mathcal{A}(N_1, N_2) \to C^1(I, I) \) as follows:

\[
Tf(t) = \frac{1}{\lambda_1(x)} F(t) - \frac{1}{\lambda_1(x)} K_f(t) + f(t), \quad \forall t, x \in I,
\]

where \( f \in \mathcal{A}(N_1, N_2) \). Because \( K_f, F, \) and \( f \) are continuously differentiable for all \( t \in I \), \( T_f \) is continuously differentiable for all \( t \in I \). By conditions (P_1) and (P_2), for any \( t_1, t_2 \) in \( I \), we have

\[
\left| \frac{d}{dt} (Tf(t)) \right| \leq \frac{1}{\lambda_1(x)} |F'(t)| + \frac{1}{\lambda_1(x)} \sum_{t=2}^{\infty} \lambda_i(x) \left| (f'(t))' \right| \leq \frac{1}{c} M_1 + \frac{1}{c} \sum_{t=2}^{\infty} c_i N_1^i
\]

\[
\leq \frac{1}{c} M_1 + \frac{1}{c} (cN_1 - M_1) = N_1.
\]

We furthermore have

\[
\left| \frac{d}{dt} (Tf(t_1)) - \frac{d}{dt} (Tf(t_2)) \right| \leq \frac{1}{\lambda_1(x)} |F'(t_1) - F'(t_2)| + \frac{1}{\lambda_1(x)} \sum_{t=2}^{\infty} c_i \left| (f'(t_1))' - (f'(t_2))' \right|
\]

\[
\leq \frac{1}{c} M_2 |t_1 - t_2| + \frac{1}{c} \sum_{t=2}^{\infty} c_i N_2 \left( \sum_{j=1}^{2i-2} N_1^j \right) |t_1 - t_2|
\]

\[
\leq N_2 |x_1 - x_2|.
\]

Thus \( T : \mathcal{A}(N_1, N_2) \to \mathcal{A}(N_1, N_2) \) is a self-diffeomorphism.

Now we prove the continuity of \( T \) under the norm \( \| \cdot \|_c \). For arbitrary \( f_1, f_2 \in \mathcal{A}(N_1, N_2) \),
\[
\|Tf_1 - Tf_2\|_\omega = \max_{i \in I} \left| -\frac{1}{\lambda_1(x)} K_{f_1}(t) + f_1(t) + \frac{1}{\lambda_1(x)} K_{f_2}(t) - f_2(t) \right|
\]
\[
\leq \frac{1}{c} \max_{i \in I} \left| \sum_{i=2}^{\infty} \lambda_i(x) f_i(t) - \sum_{i=2}^{\infty} \lambda_i(x) f_i^j(t) \right|
\]
\[
\leq \frac{1}{c} \sum_{i=2}^{\infty} c_i \left| f_i - f_i^j \right|_\omega
\]
\[
\leq \frac{1}{c} \sum_{i=2}^{\infty} c_i \left( \sum_{k=1}^{i} N_1^{k-1} \right) \left| f_i - f_i^j \right|_\omega
\]
\[
\left\| \frac{d}{dt} (Tf_1) - \frac{d}{dt} (Tf_2) \right\|_\omega = \max_{i \in I} \left| -\frac{1}{\lambda_1(x)} (K_{f_1}(t))' + (f_1(t))' + \frac{1}{\lambda_1(x)} (K_{f_2}(t))' - (f_2(t))' \right|
\]
\[
\leq \frac{1}{c} \max_{i \in I} \left| \sum_{i=2}^{\infty} \lambda_i(x) (f_i'(t))' - \sum_{i=2}^{\infty} \lambda_i(x) (f_i^j(t))' \right|
\]
\[
\leq \frac{1}{c} \sum_{i=2}^{\infty} c_i \left| (f_i')' - (f_i^j)\right|_\omega
\]
\[
\leq \frac{1}{c} \sum_{i=2}^{\infty} c_i \left[ i N_1^{i-1} \left| f_i' - f_i^j \right|_\omega + Q(i) N_2 \left( \sum_{k=1}^{i-1} (i-k) N_1^{i-k-2} \right) \left| f_i' - f_i^j \right|_\omega \right].
\] (3.10)

Let
\[
E_1 = \frac{1}{c} \sum_{i=2}^{\infty} c_i \left( \sum_{k=1}^{i} N_1^{k-1} + Q(i) N_2 \sum_{k=1}^{i-1} (i-k) N_1^{i-k-2} \right),
\]
\[
E_2 = \frac{1}{c} \sum_{i=2}^{\infty} c_i i N_1^{i-1}, \quad E = \max \{E_1, E_2\}. \quad \text{(3.11)}
\]

Then we have
\[
\left\| Tf_1 - Tf_2 \right\|_\omega = \left\| Tf_1 - Tf_2 \right\|_\omega + \left\| (Tf_1)' - (Tf_2)' \right\|_\omega \leq E_1 \left\| f_i' - f_i^j \right\|_\omega + E_2 \left\| f_i' - f_i^j \right\|_\omega
\]
\[
\leq E \left\| f_i' - f_i^j \right\|_\omega + E \left\| f_i' - f_i^j \right\|_\omega = E \left\| f_i' - f_i^j \right\|_\omega,
\] (3.12)

which gives continuity of \( T \).

It is easy to show that \( \mathcal{A}(N_1, N_2) \) is a compact convex subset of \( C^1(I, I) \). By Schauder’s fixed point theorem, we assert that there is a mapping \( f \in \mathcal{A}(N_1, N_2) \) such that
\[
f(t) = Tf(t) = \frac{1}{\lambda_1(x)} F(t) + \frac{1}{\lambda_1(x)} K_f(t) + f(t), \quad \forall t \in I.
\] (3.13)

Let \( t = x \), we have \( f(x) \) as a solution of (1.2) in \( \mathcal{A}(N_1, N_2) \). This completes the proof. \( \square \)
Theorem 3.2 (Uniqueness). Suppose that (P_1) and (P_2) are satisfied, also one supposes that

(P_3) \( E < 1, \)

then for arbitrary function \( F \) in \( \mathcal{A}(M_1, M_2) \), (1.2) has a unique solution \( f \in \mathcal{A}(N_1, N_2) \).

Proof. The existence of (1.2) in \( \mathcal{A}(N_1, N_2) \) is given by Theorem 3.1, from the proof of Theorem 3.1, we see that \( \mathcal{A}(N_1, N_2) \) is a closed subset of \( C(I, I) \), by (3.12) and (P_3), we see that \( T : \mathcal{A}(N_1, N_2) \to \mathcal{A}(N_1, N_2) \) is a contraction. Therefore \( T \) has a unique fixed point \( f(x) \) in \( \mathcal{A}(N_1, N_2) \), that is, (1.2) has a unique solution in \( \mathcal{A}(N_1, N_2) \), this proves the theorem. \( \square \)

4. Example
Consider the equation

\[
\sum_{i=1}^{\infty} \lambda_i(x) f_i(x) = \frac{1}{4} x^2, \quad x \in I := [-1, 1],
\]

(4.1)

where \( \lambda_1(x) = 33/36 + (1/36) \cos^2(\pi x/2), \lambda_2(x) = 1/36 + (1/36) \sin^2(\pi x/2), \lambda_3(x) = 1/36, \lambda_4(x) = \lambda_5(x) = \cdots = 0. \) It is easy to see that \( 0 \leq \lambda_i(x) \leq 1, \sum_{i=1}^{\infty} \lambda_i(x) = 1, c = 33/36, c_2 = 2/36, c_3 = 1/36, c_4 = c_5 = \cdots = 0. \)

For any \( x, y \) in \([-1, 1]\),

\[
|F'(x)| = |0.5x| \leq 0.5, \quad |F'(x) - F'(y)| \leq |0.5x| + |0.5y| \leq 1,
\]

(4.2)

thus \( F \in \mathcal{A}(0.5, 1) \). By condition (P_1), we can choose \( N_1 = 1.1 \), and by condition (P_1), we can choose \( N_2 = 1.5 \). Then by Theorem 3.1, there is a continuously differentiable solution of (4.1) in \( \mathcal{A}(1.1, 1.5) \).

Remark 4.1. Here \( F(x) \) is not monotone for \( x \in [-1, 1] \), hence it cannot be concluded by [11, 12].

Acknowledgments
This work was supported by Guangdong Provincial Natural Science Foundation (07301595) and Zhan-jiang Normal University Science Research Project (L0804).

References


