We give some new definitions of $D^*$-metric spaces and we prove a common fixed point theorem for a class of mappings under the condition of weakly commuting mappings in complete $D^*$-metric spaces. We get some improved versions of several fixed point theorems in complete $D^*$-metric spaces.

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1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [2], Erceg [3], Kaleva and Seikkala [4], and Kramosil and Michálek [5] have introduced the concepts of fuzzy metric spaces in different ways. George and Veeramani [6] and Kramosil and Michálek [5] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connection with both string and $E$-infinity theories which were given and studied by El Naschie [7–10]. Many authors [11–17] have studied the fixed point theory in fuzzy (probabilistic) metric spaces. On the other hand, there have been a number of generalizations of metric spaces. One of such generalizations is generalized metric space (or $D$-metric space) initiated by Dhage [18] in 1992. He proved the existence of unique fixed point of a self-map satisfying a contractive condition in complete and bounded $D$-metric spaces. Dealing with $D$-metric space, Ahmad et al. [19], Dhage [18, 20], Dhage et al. [21], Rhoades [22], Singh and Sharma [23], and others made a significant contribution in fixed point theory of $D$-metric space. Unfortunately, almost all theorems in $D$-metric spaces are not valid (see [24–26]).
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In this paper, we introduce \( D^* \)-metric which is a probable modification of the definition of \( D \)-metric introduced by Dhage [18, 20] and prove some basic properties in \( D^* \)-metric spaces.

In what follows \((X,D^*)\) will denote a \( D^* \)-metric space, \( \mathbb{N} \) the set of all natural numbers, and \( \mathbb{R}^+ \) the set of all positive real numbers.

**Definition 1.1.** Let \( X \) be a nonempty set. A generalized metric (or \( D^* \)-metric) on \( X \) is a function, \( D^* : X^3 \rightarrow [0, \infty) \), that satisfies the following conditions for each \( x, y, z, a \in X \):

1. \( D^*(x, y, z) \geq 0 \),
2. \( D^*(x, y, z) = 0 \) if and only if \( x = y = z \),
3. \( D^*(x, y, z) = D^*(p \{x, y, z\}) \), (symmetry) where \( p \) is a permutation function,
4. \( D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, a) \).

The pair \((X,D^*)\) is called a generalized metric (or \( D^* \)-metric) space.

Immediate examples of such a function are

(a) \( D^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\} \),
(b) \( D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x) \).

Here, \( d \) is the ordinary metric on \( X \).

(c) If \( X = \mathbb{R}^n \) then we define

\[
D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p} \tag{1.1}
\]

for every \( p \in \mathbb{R}^+ \).

(d) If \( X = \mathbb{R} \), then we define

\[
D^*(x, y, z) = \begin{cases} 
0 & \text{if } x = y = z, \\
\max \{x, y, z\} & \text{otherwise.} 
\end{cases} \tag{1.2}
\]

**Remark 1.2.** In a \( D^* \)-metric space, we prove that \( D^*(x, x, y) = D^*(x, y, y) \). For

(i) \( D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y) \) and similarly

(ii) \( D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x) \).

Hence by (i), (ii) we get \( D^*(x, x, y) = D^*(x, y, y) \).

Let \((X,D^*)\) be a \( D^* \)-metric space. For \( r > 0 \), define

\[
B_{D^*}(x, r) = \{ y \in X : D^*(x, y, y) < r \}. \tag{1.3}
\]

**Example 1.3.** Let \( X = \mathbb{R} \). Denote \( D^*(x, y, z) = |x - y| + |y - z| + |z - x| \) for all \( x, y, z \in \mathbb{R} \). Thus

\[
B_{D^*}(1, 2) = \{ y \in \mathbb{R} : D^*(1, y, y) < 2 \}
= \{ y \in \mathbb{R} : |y - 1| + |y - 1| < 2 \}
= \{ y \in \mathbb{R} : |y - 1| < 1 \} = (0, 2). \tag{1.4}
\]
Lemma 1.4. Let \((X,D^*)\) be a \(D^*\)-metric space and \(A \subset X\).

(1) If for every \(x \in A\), there exists \(r > 0\) such that \(B_{D^*}(x,r) \subset A\), then subset \(A\) is called open subset of \(X\).

(2) Subset \(A\) of \(X\) is said to be \(D^*\)-bounded if there exists \(r > 0\) such that \(D^*(x,y) < r\) for all \(x,y \in A\).

(3) A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if \(D^*(x_n,x_m,x) = D^*(x,x,x_n) \to 0\) as \(n \to \infty\). That is, for each \(\varepsilon > 0\) there exists \(n_0 \in \mathbb{N}\) such that

\[\forall n \geq n_0 \implies D^*(x,x,x_n) < \varepsilon(\ast).\]  

This is equivalent; for each \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[\forall n,m \geq n_0 \implies D^*(x,x_n,x_m) < \varepsilon(\ast\ast).\]  

Indeed, if \((\ast)\) holds, then

\[D^*(x_n,x_m,x) = D^*(x_n,x_m,x) \leq D^*(x_n,x,x) + D^*(x,x_m,x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\]  

Conversely, set \(m = n\) in \((\ast\ast)\), then we have \(D^*(x_n,x_n,x) < \varepsilon\).

(4) A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if for each \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(D^*(x_n,x_n,x_m) < \varepsilon\) for each \(n,m \geq n_0\). The \(D^*\)-metric space \((X,D^*)\) is said to be complete if every Cauchy sequence is convergent.

Let \(\tau\) be the set of all \(A \subset X\) with \(x \in A\) if and only if there exists \(r > 0\) such that \(B_{D^*}(x,r) \subset A\). Then \(\tau\) is a topology on \(X\) (induced by the \(D^*\)-metric \(D^*\)).

**Lemma 1.5.** Let \((X,D^*)\) be a \(D^*\)-metric space. If \(r > 0\), then ball \(B_{D^*}(x,r)\) with center \(x \in X\) and radius \(r\) is open ball.

**Proof.** Let \(z \in B_{D^*}(x,r)\), hence \(D^*(x,z,z) < r\). Let \(D^*(x,z,z) = \delta \) and \(r' = r - \delta\). Let \(y \in B_{D^*}(z,r')\), by triangular inequality we have \(D^*(x,y,y) = D^*(y,y,x) \leq D^*(y,y,z) + D^*(z,x,x) < r' + \delta = r\). Hence \(B_{D^*}(z,r') \subseteq B_{D^*}(x,r)\). Hence the ball \(B_{D^*}(x,r)\) is an open ball.

**Definition 1.6.** Let \((X,D^*)\) be a \(D^*\)-metric space. \(D^*\) is said to be a continuous function on \(X^3\) if

\[\lim_{n \to \infty} D^*(x_n,y_n,z_n) = D^*(x,y,z)\]  

whenever a sequence \(\{(x_n,y_n,z_n)\}\) in \(X^3\) converges to a point \((x,y,z) \in X^3\), that is,

\[\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z.\]  

**Lemma 1.7.** Let \((X,D^*)\) be a \(D^*\)-metric space. Then \(D^*\) is a continuous function on \(X^3\).

**Proof.** Suppose the sequence \(\{(x_n,y_n,z_n)\}\) in \(X^3\) converges to a point \((x,y,z) \in X^3\), that is,

\[\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z.\]  

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Then for each $\epsilon > 0$ there exist $n_1$, $n_2$, and $n_3 \in \mathbb{N}$ such that $D^*(x,x,x_n) < \epsilon/3 \forall n \geq n_1$, $D^*(y,y,y_n) < \epsilon/3$ for all $n \geq n_2$, and $D^*(z,z,z_n) < \epsilon/3 \forall n \geq n_3$.

If we set $n_0 = \max \{n_1,n_2,n_3\}$, then for all $n \geq n_0$ by triangular inequality we have

\[
D^*(x_n,y_n,z_n) \leq D^*(x_n,y_n,y) + D^*(y,y_n,z_n) \\
\leq D^*(x,z,y) + D^*(y,y_n) + D^*(z,z_n,z_n) \\
\leq D^*(z,y,x) + D^*(x,x_n,x_n) + D^*(y,y_n,y_n) + D^*(z,z_n,z_n) \\
< D^*(x,y,z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x,y,z) + \epsilon.
\]

Hence we have

\[
D^*(x_n,y_n,z_n) \leq D^*(x,y,z) + \epsilon < \epsilon,
\]

\[
D^*(x,y,z) \leq D^*(x,y,z_n) + D^*(z_n,z,z) \\
\leq D^*(x,z_n,y_n) + D^*(y_n,y,y) + D^*(z,z,z) \\
\leq D^*(z,y_n,x_n) + D^*(x_n,x,x) + D^*(y_n,y,y) + D^*(z,z,z) \\
\leq D^*(x_n,y_n,z_n) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x_n,y_n,z_n) + \epsilon.
\]

That is,

\[
D^*(x,y,z) - D^*(x_n,y_n,z_n) < \epsilon.
\]

Therefore we have $|D^*(x_n,y_n,z_n) - D^*(x,y,z)| < \epsilon$, that is,

\[
\lim_{n \to \infty} D^*(x_n,y_n,z_n) = D^*(x,y,z).
\]

**Lemma 1.8.** Let $(X,D^*)$ be a $D^*$-metric space. If sequence $\{x_n\}$ in $X$ converges to $x$, then $x$ is unique.

**Proof.** Let $x_n \to y$ and $y \neq x$. Since $\{x_n\}$ converges to $x$ and $y$, for each $\epsilon > 0$ there exist $n_1,n_2 \in \mathbb{N}$ such that $D^*(x,x,x_n) < \epsilon/2 \forall n \geq n_1$ and $D^*(y,y,y_n) < \epsilon/2 \forall n \geq n_2$.

If we set $n_0 = \max \{n_1,n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

\[
D^*(x,x,y) \leq D^*(x,x,x_n) + D^*(x_n,y,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Hence $D^*(x,x,y) = 0$ which is a contradiction. So, $x = y$. **□**

**Lemma 1.9.** Let $(X,D^*)$ be a $D^*$-metric space. If sequence $\{x_n\}$ in $X$ is convergent to $x$, then sequence $\{x_n\}$ is a Cauchy sequence.
Proof. Since \( x_n \to x \), for each \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( D^*(x_n,x_n,x) < \epsilon/2 \forall n \geq n_0 \). Then for every \( n,m \geq n_0 \), by triangular inequality, we have

\[
D^*(x_n,x_n,x_m) \leq D^*(x_n,x_n,x) + D^*(x,x,x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Hence sequence \( \{x_n\} \) is a Cauchy sequence. \( \square \)

Definition 1.10. Let \( A \) and \( S \) be two mappings from a \( D^* \)-metric space \((X,D^*)\) into itself. Then \( \{A,S\} \) is said to be weakly commuting pair if

\[
D^*(ASx,SAx,SAx) \leq D^*(Ax,Sx,Sx),
\]

for all \( x \in X \). Clearly, a commuting pair is weakly commuting, but not conversely as shown in the following example.

Example 1.11. Let \((X,D^*)\) be a \( D^* \)-metric space, where \( X = [0, 1] \) and

\[
D^*(x,y,z) = |x - y| + |y - z| + |x - z|.
\]

Define self-maps \( A \) and \( S \) on \( X \) as follows:

\[
Sx = \frac{x}{2}, \quad Ax = \frac{x}{x + 2} \quad \forall x \in X.
\]

Then for all \( x \in X \) one gets

\[
D^*(SAx,ASx,ASx) = \left| \frac{x}{x + 4} - \frac{x}{2x + 4} \right| + \left| \frac{x}{x + 4} - \frac{x}{x + 4} \right| + \left| \frac{x}{x + 4} - \frac{x}{2x + 4} \right|
\]

\[
= \frac{2x^2}{(x + 4)(2x + 4)} \leq \frac{2x^2}{2x + 4}
\]

\[
= \left| \frac{x}{2} - \frac{x}{x + 2} \right| + \left| \frac{x}{2} - \frac{x}{x + 2} \right| + 0
\]

\[
= D^*(Sx,Ax,Ax).
\]

So \( \{A,S\} \) is a weakly commuting pair.

However, for any nonzero \( x \in X \) we have

\[
SAx = \frac{x}{x + 4} > \frac{x}{2x + 4} = ASx.
\]

Thus \( A \) and \( S \) are not commuting mappings.

2. The main results

A class of implicit relation. Throughout this section \((X,D^*)\) denotes a \( D^* \)-metric space and \( \Phi \) denotes a family of mappings such that each \( \varphi \in \Phi \), \( \varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+ \), and \( \varphi \) is continuous and increasing in each coordinate variable. Also \( \gamma(t) = \varphi(t,t,a_1t,a_2t,t) < t \) for every \( t \in \mathbb{R}^+ \) where \( a_1 + a_2 = 3 \).
Example 2.1. Let \( \varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+ \) be defined by
\[
\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{7} (t_1 + t_2 + t_3 + t_4 + t_5).
\]  

The following lemma is the key in proving our result.

Lemma 2.2. For every \( t > 0 \), \( \gamma(t) < t \) if and only if \( \lim_{n \to \infty} \gamma^n(t) = 0 \), where \( \gamma^n \) denotes the composition of \( \gamma \) with itself \( n \) times.

Our main result, for a complete \( D^* \)-metric space \( X \), reads as follows.

Theorem 2.3. Let \( A \) be a self-mapping of complete \( D^* \)-metric space \((X, D^*)\), and let \( S, T \) be continuous self-mappings on \( X \) satisfying the following conditions:
(i) \( \{A, S\} \) and \( \{A, T\} \) are weakly commuting pairs such that \( A(X) \subseteq S(X) \cap T(X) \);
(ii) there exists a \( \varphi \in \Phi \) such that for all \( x, y \in X \),
\[
D^*(Ax, Ay, Az) \\
\leq \varphi(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)).
\]  

Then \( A, S, \) and \( T \) have a unique common fixed point in \( X \).

Proof. Let \( x_0 \in X \) be an arbitrary point in \( X \). Then \( Ax_0 \in X \). Since \( A(X) \) is contained in \( S(X) \), there exists a point \( x_1 \in X \) such that \( Ax_0 = Sx_1 \). Since \( A(X) \) is also contained in \( T(X) \), we can choose a point \( x_2 \in X \) such that \( Ax_1 = Tx_2 \). Continuing this way, we define by induction a sequence \( \{x_n\} \) in \( X \) such that
\[
Sx_{2n+1} = Ax_{2n} = y_{2n}, \quad n = 0, 1, 2, \ldots,
\]
\[
Tx_{2n+2} = Ax_{2n+1} = y_{2n+1}, \quad n = 0, 1, 2, \ldots.
\]  

For simplicity, we set
\[
d_n = D^*(y_{2n}, y_{2n+1}, y_{2n+1}), \quad n = 0, 1, 2, \ldots.
\]  

We prove that \( d_{2n} \leq d_{2n-1} \). Now, if \( d_{2n} > d_{2n-1} \) for some \( n \in \mathbb{N} \), since \( \varphi \) is an increasing function, then
\[
d_{2n} = D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = D^*(Ax_{2n}, Ax_{2n+1}, Ax_{2n+1}) = D^*(Ax_{2n+1}, Ax_{2n}, Ax_{2n})
\]
\[
\leq \varphi\left( \begin{array}{c}
D^*(Sx_{2n+1}, Tx_{2n}, Tx_{2n}), \\
D^*(Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}), \\
D^*(Tx_{2n}, Ax_{2n+1}, Ax_{2n+1})
\end{array} \right) = \varphi\left( \begin{array}{c}
D^*(y_{2n}, y_{2n-1}, y_{2n-1}), \\
D^*(y_{2n}, y_{2n+1}, y_{2n+1}), \\
D^*(y_{2n}, y_{2n}, y_{2n})
\end{array} \right).
\]  

(2.5)
Since

\[ D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}) \leq D^*(y_{2n-1}, y_{2n-1}, y_{2n}) + D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = d_{2n-1} + d_{2n}, \]

(2.6)

hence by the above inequality we have

\[ d_{2n} \leq \varphi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) \leq \varphi(d_{2n}, d_{2n}, 2d_{2n}, d_{2n}) < d_{2n}, \]

(2.7)
a contradiction. Hence \( d_{2n} \leq d_{2n-1} \). Similarly, one can prove that \( d_{2n+1} \leq d_{2n} \) for \( n = 0, 1, 2, \ldots \). Consequently, \( \{ d_n \} \) is a nonincreasing sequence of nonnegative reals. Now,

\[
d_1 = D^*(y_1, y_2, y_3) = D^*(Ax_1, Ax_2, Ax_3)
\]

\[
\leq \varphi \left( D^*(Sx_1, Tx_2, Tx_2), D^*(Sx_1, Ax_1, Ax_1), D^*(Sx_1, Ax_2, Ax_2), D^*(Tx_2, Ax_2, Ax_2) \right)
\]

\[
= \varphi \left( D^*(y_0, y_1, y_1), D^*(y_0, y_1, y_1), D^*(y_0, y_2, y_2) \right)
\]

(2.8)

\[
= \varphi(d_0, d_0, d_0 + d_1, 0, d_0)
\]

\[
\leq \varphi(d_0, d_0, 2d_0, d_0, d_0) = \gamma(y_0).
\]

In general, we have \( d_n \leq \gamma^n(d_0) \). So if \( d_0 > 0 \), then Lemma 2.2 gives \( \lim_{n \to \infty} d_n = 0 \).

For \( d_0 = 0 \), we clearly have \( \lim_{n \to \infty} d_n = 0 \), since then \( d_n = 0 \) for each \( n \). Now we prove that sequence \( \{ Ax_n = y_n \} \) is a Cauchy sequence. Since \( \lim_{n \to \infty} d_n = 0 \), it is sufficient to show that the sequence \( \{ Ax_{2n} = y_{2n} \} \) is a Cauchy sequence. Suppose that \( \{ Ax_{2n} = y_{2n} \} \) is not a Cauchy sequence. Then there is an \( \epsilon > 0 \) such that for each even integer \( 2k \), for \( k = 0, 1, 2, \ldots \), there exist even integers \( 2n(k) \) and \( 2m(k) \) with \( 2k \leq 2n(k) < 2m(k) \) such that

\[ D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \]

(2.9)

Let, for each even integer \( 2k, 2m(k) \) be the least integer exceeding \( 2n(k) \) satisfying (2.9). Therefore

\[ D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)} - 2) \leq \epsilon, \quad D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \]

(2.10)

Then, for each even integer \( 2k \) we have

\[
\epsilon < D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)})
\]

\[
\leq D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)} - 2) + D^*(Ax_{2m(k)} - 2, Ax_{2m(k)} - 2, Ax_{2m(k)} - 1)
\]

\[
+ D^*(Ax_{2m(k)} - 1, Ax_{2m(k)} - 1, Ax_{2m(k)})
\]

(2.11)

\[
= D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)} - 2) + d_{2m(k) - 2} + d_{2m(k) - 1}.
\]
They have the same limit $z$.

Using (2.14),

$$\lim_{k \to \infty} D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) = \epsilon.$$  \hspace{1cm} \text{(2.12)}

It follows immediately from the triangular inequality that

$$\left| D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)} - 1) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \right| \leq d_{2m(k) - 1},$$

$$\left| D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \right| < d_{2m(k) - 1} + d_{2m(k)}.$$

Hence by (2.10), as $k \to \infty$,

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k) - 1}) \to \epsilon,$$

$$D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k) - 1}) \to \epsilon.$$  \hspace{1cm} \text{(2.14)}

Now

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \leq D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2n(k)+1}) + D^*(Ax_{2n(k)+1}, Ax_{2m(k)}, Ax_{2m(k)})$$

$$\leq d_{2n(k)} + \phi \left( D^*(Ax_{2n(k)}, Ax_{2m(k) - 1}, Ax_{2m(k) - 1}), d_{2n(k)} \right) + \phi \left( D^*(Ax_{2n(k)+1}, Ax_{2m(k)+1}, Ax_{2m(k)+1}), d_{2m(k) - 1} \right).$$  \hspace{1cm} \text{(2.15)}

Using (2.14), $\lim_{k \to \infty} d_n = 0$, and continuity and nondecreasing property of $\phi$ in each coordinate variable, we have

$$\epsilon \leq \phi(\epsilon, 0, \epsilon, 0) \leq \phi(\epsilon, \epsilon, 2\epsilon, \epsilon, \epsilon) = \gamma(\epsilon) < \epsilon$$  \hspace{1cm} \text{(2.16)}

as $k \to \infty$, which is a contradiction. Thus $\{Ax_n = y_n\}$ is a Cauchy sequence and hence by completeness of $X$, it converges to $z \in X$. That is,

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} y_n = z.$$  \hspace{1cm} \text{(2.17)}

Since the sequences $\{Sx_{2n+1} = y_{2n+1}\}$ and $\{Tx_{2n} = y_{2n}\}$ are subsequences of $\{Ax_n = y_n\}$; they have the same limit $z$. As $S$ and $T$ are continuous, we have $STx_{2n} \to Sz$ and $TSx_{2n+1} \to Tz$.

Now consider

$$D^*(STx_{2n}, Tsx_{2n+1}, Tsx_{2n+1}) = D^*(SAx_{2n-1}, TAx_{2n}, TAx_{2n})$$

$$\leq D^*(SAx_{2n-1}, Asx_{2n-1}, Asx_{2n-1})$$

$$+ D^*(Asx_{2n-1}, Asx_{2n-1}, ATx_{2n})$$

$$+ D^*(ATx_{2n}, ATx_{2n}, TAx_{2n}).$$  \hspace{1cm} \text{(2.18)}
Using (ii) and the weak commutativity of \( \{A, S\} \) and \( \{A, T\} \), we get

\[
D^* (STx_{2n}, TSx_{2n+1}, TSx_{2n+1}) \\
\leq D^* (Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) + D^* (ASx_{2n-1}, ATx_{2n}, ATx_{2n}) + D^* (Ax_{2n}, Ax_{2n}, Tz_{2n}) \\
\leq D^* (Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \\
+ \varphi \left( \begin{array}{c}
D^* (S^2x_{2n-1}, T^2x_{2n}, T^2x_{2n}), \\
D^* (S^2x_{2n-1}, AXx_{2n-1}, AXx_{2n-1}), \\
D^* (T^2x_{2n}, AXx_{2n-1}, AXx_{2n-1}), \\
D^* (T^2x_{2n}, TXx_{2n}, TXx_{2n}) + D^* (Ax_{2n}, Ax_{2n}, Ax_{2n})
\end{array} \right)
\]

\[
\leq D^* (Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \\
+ \varphi \left( \begin{array}{c}
D^* (S^2x_{2n-1}, T^2x_{2n}, T^2x_{2n}), \\
D^* (S^2x_{2n-1}, AXx_{2n-1}, AXx_{2n-1}), \\
D^* (T^2x_{2n}, AXx_{2n-1}, AXx_{2n-1}), \\
D^* (T^2x_{2n}, TXx_{2n}, TXx_{2n}) + D^* (Ax_{2n}, Ax_{2n}, Ax_{2n})
\end{array} \right)
\]

\[
\leq D^* (Ax_{2n}, Ax_{2n}, Tz_{2n}). \\
\] (2.19)

If \( D^* (Sz, Tz, Tz) > 0 \), then as \( n \to \infty \) we have

\[
D^* (Sz, Tz, Tz) \\
\leq D^* (z, z, z) + \varphi \left( \begin{array}{c}
D^* (Sz, Tz, Tz), \\
D^* (Sz, Sz, Sz) + 0, \\
D^* (Tz, Sz, Sz) + 0
\end{array} \right) + 0 \\
\leq \gamma (D^*(Sz, Tz, Tz)) < D^*(Sz, Tz, Tz), \\
\] (2.20)

a contradiction. Therefore, \( Sz = Tz \).

Now we will prove that \( Az = Sz \). To end this, consider the inequality

\[
D^* (SAx_{2n+1}, Az, Az) \leq D^* (SAx_{2n+1}, AXx_{2n+1}, AXx_{2n+1}) + D^* (Az, Az, ASx_{2n+1}). \\
\] (2.21)

Again using (ii) and the weak commutativity of \( \{A, S\} \), we have

\[
D^* (SAx_{2n+1}, Az, Az) \leq D^* (Ax_{2n+1}, AXx_{2n+1}, Ax_{2n+1}) \\
+ \varphi \left( \begin{array}{c}
D^* (Sz, Tz, TSx_{2n+1}), \\
D^* (Sz, Az, Az), D^* (Sz, Az, Az)
\end{array} \right)
\]

\[
+ \varphi \left( \begin{array}{c}
D^* (Tz, Az, Az) \\
D^* (Tz, Az, Az)
\end{array} \right). \\
\] (2.22)
Taking $n \to \infty$, we have
\[
D^*(Sz, Az, Az) \leq D^*(z, z, z) + \varphi \left( \frac{D^*(Sz, Tz, Tz)}{D^*(Tz, Az, Az)}, \frac{D^*(Sz, Az, Az)}{D^*(Tz, Az, Az)} \right)
\]
\[
= \varphi(0, D^*(Sz, Az, Az), D^*(Sz, Az, Az), D^*(Sz, Az, Az))
\]
\[
\leq \delta(D^*(Sz, Az, Az)) < D^*(Sz, Az, Az)
\]
(2.23)
given there by $Sz = Az$. Thus $Az = Sz = Tz$. It now follows that
\[
D^*(Az, Ax_{2n}, Ax_{2n}) \leq \varphi \left( \frac{D^*(Sz, Tx_{2n}, Tx_{2n})}{D^*(Tx_{2n}, Az, Az)}, \frac{D^*(Sz, Ax_{2n}, Ax_{2n})}{D^*(Tx_{2n}, Ax_{2n}, Ax_{2n})} \right).
\]
(2.24)
Then as $n \to \infty$, we get
\[
D^*(Az, z, z) \leq \varphi(D^*(Sz, z, z), 0, D^*(Sz, z, z), D^*(z, Az, Az), 0)
\]
\[
\leq \gamma(D^*(Az, z, z)) < D^*(Az, z, z),
\]
(2.25)
a contradiction, and therefore $Az = z = Sz = Tz$. Thus $z$ is a common fixed point of $A, S,$ and $T$. The unicity of the common fixed point is not hard to verify. This completes the proof of the theorem. \(\square\)

Example 2.4. Let $(X, D^*)$ be a $D^*$-metric space, where $X = [0, 1]$ and
\[
D^*(x, y, z) = |x - y| + |y - z| + |x - z|.
\]
(2.26)
Define self-maps $A, T,$ and $S$ on $X$ as follows:
\[
Sx = x, \quad Ax = 1, \quad Tx = \frac{x + 1}{2},
\]
(2.27)
for all $x \in X$.
Let
\[
\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{7}(t_1 + t_2 + t_3 + t_4 + t_5).
\]
(2.28)
Then
\[
A(X) = \{1\} \subset [0, 1] \cap \left[ \frac{1}{2}, 1 \right] = S(X) \cap T(X),
\]
(2.29)
and for every $x \in X$, we have
\[
D^*(ATx, TAx, TAx) = D^*(1, 1, 1) = 0 \leq D^*(Ax, Tx, Tx),
\]
\[
D^*(ASx, SAx, SAx) = D^*(1, 1, 1) = 0 \leq D^*(Ax, Sx, Sx).
\]
(2.30)
That is, the pairs $(A, S)$ and $(A, T)$ are weakly commuting.
Also for all \(x, y, z \in X\), we have
\[
D^*(Ax,Ay,Az) = 0
\]
\[
\leq \varphi(D^*(Sx,Ty,Tz),D^*(Ax,Ax),D^*(Ay,Ay),D^*(Ty,Ax,Ax),D^*(Ty,Ay,Ay)).
\]
(2.31)

That is, all conditions of Theorem 2.3 hold and 1 is the unique common fixed point of
\(A, S, \text{and } T\).

**Corollary 2.5.** Let \(A,R,S,T, \text{and } H\) be self-mappings of complete \(D^*-\text{metric space } (X, D^*)\), and let \(SR,TH\) be continuous self-mappings on \(X\) satisfying the following conditions:

(i) \(\{A,SR\} \text{ and } \{A,TH\}\) are weakly commuting pairs such that \(A(X) \subset SR(X) \cap TH(X)\);

(ii) there exists a \(\varphi \in \Phi\) such that for all \(x, y \in X\),

\[
D^*(Ax,Ay,Az) \leq \varphi \left( D^*(SRx,THy,THz), D^*(SRx,Ax,Ax), D^*(SRx,Ay,Ay), \right.
\]
\[
\left. D^*(THy,Ax,Ax), D^*(THy,Ay,Ay) \right).
\]
(2.32)

If \(SR = RS, TH = HT, AH = HA, \text{and } AR = RA\), then \(A, S, R, H, \text{and } T\) have a unique common fixed point in \(X\).

**Proof.** By Theorem 2.3, \(A, TH, \text{and } SR\) have a unique common fixed point in \(X\). That is, there exists \(a \in X\), such that \(A(a) = TH(a) = SR(a) = a\). We prove that \(R(a) = a\). By (ii), we get
\[
D^*(ARa,Aa,Aa) \leq \varphi \left( D^*(SRRa,THa,THa), D^*(SRRa,ARa,ARa), D^*(SRRa,Aa,Aa), \right.
\]
\[
\left. D^*(THa,ARa,ARa), D^*(THa,Aa,Aa) \right).
\]
(2.33)

Hence if \(Ra \neq a\), then we have
\[
D^*(Ra,a,a) \leq \varphi(D^*(Ra,a,a),D^*(Ra,Ra,Ra),D^*(Ra,a,a),D^*(Ra,Ra,Ra),D^*(a,a,a))
\]
\[
\leq \varphi(D^*(Ra,a,a),D^*(Ra,a,a),D^*(Ra,a,a), 2D^*(Ra,a,a),D^*(Ra,a,a))
\]
\[
< D^*(Ra,a,a),
\]
(2.34)
a contradiction. Therefore it follows that \(Ra = a\). Hence \(S(a) = SR(a) = a\). Similarly, we get that \(T(a) = H(a) = a\). \(\square\)

**Corollary 2.6.** Let \(A_i\) be a sequence self-mapping of complete \(D^*-\text{metric space } (X, D^*)\) for \(i \in \mathbb{N}\), and let \(S, T\) be continuous self-mappings on \(X\) satisfying the following conditions:

(i) there exists \(i_0 \in \mathbb{N}\) such that \(\{A_{i_0},S\} \text{ and } \{A_{i_0},T\}\) are weakly commuting pairs such that \(A_{i_0}(X) \subset S(X) \cap T(X)\);
Then $A_i, S,$ and $T$ have a unique common fixed point in $X$ for every $i \in \mathbb{N}$.

**Proof.** By Theorem 2.3, $S$, $T$, and $A_{i_0}$, for some $i = j = k = i_0 \in \mathbb{N}$, have a unique common fixed point in $X$. That is, there exists a unique $a \in X$ such that

$$S(a) = T(a) = A_{i_0}(a) = a. \quad (2.36)$$

Suppose there exists $i \in \mathbb{N}$ such that $i \neq i_0$ and $j = i_0, k = i_0$. Then we have

$$D^*(A_i a, A_{i_0} a, A_{i_0} a) \leq \varphi \left( D^*(S a, T a, T a), D^*(S a, A_i a, A_i a), D^*(S a, A_{i_0} a, A_{i_0} a), \right. \left. \frac{D^*(T a, A_i a, A_i a)}{D^*(T a, A_{i_0} a, A_{i_0} a)} \right). \quad (2.37)$$

Hence if $A_i a \neq a$, then we get

$$D^*(A_i a, a, a, a) \leq \varphi \left( D^*(a, a, a), D^*(a, A_i a, A_i a), D^*(a, a, a), \right. \left. \frac{D^*(A_i a, a, a)}{D^*(A_i a, a, a)} \right. \left. \frac{D^*(A_i a, a, a)}{D^*(A_i a, a, a)} \right) \leq \varphi \left( D^*(A_i a, a, a), D^*(A_i a, a, a), D^*(A_i a, a, a), \right. \left. \frac{2D^*(A_i a, a, a)}{D^*(A_i a, a, a)} \right) \quad (2.38)$$

a contradiction. Hence for every $i \in \mathbb{N}$ it follows that $A_i(a) = a$ for every $i \in \mathbb{N}$. \hfill \Box

**References**


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