Research Article

Oscillatory Solutions of Neutral Equations with Polynomial Nonlinearities

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Existence uniqueness of an oscillatory solution for nonlinear neutral equations by fixed point method is proved.

1. Introduction

In [1, 2], we have considered a lossless transmission line terminated by a nonlinear resistive load and parallel connected capacitance (cf. Figure 1). The nonlinear boundary condition is caused by the polynomial type V-I characteristics of the nonlinear load at the second end of the transmission line (cf. Figure 1).

The voltage and current \( u(x,t), \ i(x,t) \) of the lossless transmission line can be found by solving the following mixed problem for the hyperbolic partial differential system:

\[
C \frac{\partial u(x,t)}{\partial t} + \frac{\partial i(x,t)}{\partial x} = 0, \quad L \frac{\partial i(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = 0,
\]

\( E(t) - u(0,t) = R_0 i(0,t), \quad t \geq 0, \)  

(1.1)

\[
C_0 \frac{du(\Lambda,t)}{dt} = i(\Lambda,t) - f(u(\Lambda,t)), \quad t \geq 0,
\]

(1.2)

\[
u(x,0) = u_0(x), \quad i(x,0) = i_0(x), \quad x \in [0, \Lambda],
\]

(1.3)
where \( u_0(x) \) and \( i_0(x) \) are prescribed initial functions, \( \Lambda \) is the length of the line, \( C \) is the per-unit length capacitance, and \( L \) is per-unit length inductance (cf. [3–10]). Here, the V-I characteristic of the nonlinear resistive load is \( i = f(u) = \sum_{n=1}^{p} r_n u^n \), where \( r_n \) are real numbers, \( C_0 \) is parallel connected capacitance, \( E \) is the source voltage, \( R_0 \) is the source resistance, and \( Z_0 = \sqrt{L/C} \) is the line characteristic impedance.

The above formulated mixed problem can be reduced (cf. [1, 2, 11]) to an equivalent initial value problem for a neutral functional differential equation (cf. [12]). Here, we consider the problem of an existence uniqueness of oscillatory solutions of the equation

\[
\frac{du(t)}{dt} = \frac{2E}{C_0(Z_0 + R_0)} - \frac{u(t)}{C_0 Z_0} - \frac{1}{C_0} \sum_{n=1}^{p} r_n [u(t)]^n - \frac{(Z_0 - R_0) u(t - 2T)}{Z_0 C_0 (Z_0 + R_0)} + \frac{Z_0 - R_0}{C_0 (Z_0 + R_0)} \sum_{n=1}^{p} r_n [u(t - 2T)]^n + \frac{Z_0 - R_0}{Z_0 + R_0} \frac{du(t - 2T)}{dt}, \quad t \geq T,
\]

\[
u(t) = \nu_0(t), \quad \frac{du(t)}{dt} = \frac{d\nu_0(t)}{dt}, \quad t \in [-T, T],
\]

where \((x, t) \in \Pi = \{(x, t) \in \mathbb{R}^2 : (x, t) \in [0, \Lambda] \times [0, \infty)\}, \kappa = \frac{|Z_0 - R_0|}{(Z_0 + R_0)} < 1, \nu(t) = u(\Lambda, t).\) In fact, (1.4) is differential difference equation, and the initial function should be prescribed on an interval with length \(2T\). Let us note that the initial function \( \nu_0(t) \) can be obtained shifting the initial function \( u_0(x) \) from (1.3) along the characteristics \( x - vt = \text{const.} \) (\( v = 1/\sqrt{LC} \)) on \([0, T]\) and along the characteristics \( x + vt = \text{const.} \) on \([-T, 0]\) (cf. [1, 2]). So, we obtain an initial function \( \nu_0(t) \) on \([-T, T]\).

Now, we are able to formulate the main problem: to find a solution of (1.4) with advanced prescribed zeros on the interval \([t_0, \infty)\), \( T = t_0 \).

Let \( S_T = \{\tau_k\}_{k=0}^{n}, n \in \mathbb{N} \) be the set of zeros of the initial function; that is, \( \nu_0(\tau_k) = 0 \) such that \( \tau_0 = -T, \tau_n = T = t_0 \).

Let \( S = \{t_k\}_{k=0}^{\infty} \) be a strictly increasing sequence of real numbers satisfying the following conditions (C):

(1) \( \lim_{k \to \infty} t_k = \infty \),

(2) \( 0 < t_0 = \inf\{t_{k+1} - t_k : k = 0, 1, 2, \ldots\} \leq \sup\{t_{k+1} - t_k : k = 0, 1, 2, \ldots\} = T_0 < \infty \),

(3) for every \( k \) there is \( s < k \) such that \( t_k - T = t_s \) where \( t_s \in S_T \cup S \).

Introduce the sets: \( C^1[t_0, \infty) \) consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval \( (t_k, t_{k+1}) \) (the derivatives at \( t_k \) do
not necessary exist), \( M_S = \{ u(\cdot) \in C^1[0, \infty] : u(t_k) = 0 \ (k = 0, 1, 2, \ldots) \} \), \( M_{SLU} = \{ u(\cdot) \in M_S : |u(t)| \leq U_0 e^{\mu(t-t_k)}, \ t \in [t_k, t_{k+1}] \} \), where \( U_0, \mu \) are positive constants prescribed below.

We assume that \( |o_0(t)| \leq U_0 e^{\mu(t-t_k)}, \ t \in [t_k, t_{k+1}] \), \( k = 0, 1, 2, \ldots, n-1 \).

The set \( M_{SLU} \) turns out into a complete uniform space with respect to the family of pseudometrics \( \rho_{\mu}^{(k)}(f, g) = \max\{|\rho_k(f, g), \rho_k(f, g)\}, \ (k = 0, 1, 2, \ldots), \text{where} \rho_k(f, g) = \max\{e^{-\mu(t-t_k)}|f(t) - g(t)| : t \in [t_k, t_{k+1}]\}, \rho_k(f, g) = \max\{e^{-\mu(t-t_k)}|f(t) - g(t)| : t \in [t_k, t_{k+1}]\} \).

One can verify that \( M_{SLU} \) is closed subset of \( C^1[0, \infty) \) with respect to the above metric.

**Remark 1.1.** The functions from \( M_S \) are not necessary differentiable at \( t_k \ (k = 0, 1, 2, \ldots) \). That is why we consider a space with a countable family of pseudometrics, and then, we have to apply the fixed point theory from [13].

Define the operator \( B : M_{SLU} \rightarrow M_{SLU} \) by

\[
B(u)(t) : = \int_{t_k}^{t} U(u(s))ds - \left( \frac{t-t_k}{t_{k+1}-t_k} \right) \int_{t_k}^{t_{k+1}} U(u(s))ds, \quad t \in [t_k, t_{k+1}], \ (k = 0, 1, 2, \ldots),
\]

(1.5)

where

\[
U(u)(t) = \frac{2E}{C_0(Z_0 + R_0)} - \frac{u(t)}{C_0Z_0} - \frac{1}{C_0} \sum_{n=1}^{p} r_n[u(t)]^n - \frac{\kappa(K_T u)(t)}{Z_0C_0} + \frac{\kappa}{C_0} \sum_{n=1}^{p} r_n[(K_T u)(t)]^n + \kappa \frac{d(K_T u)(t)}{dt}, \quad t \geq T,
\]

(1.6)

and \( (K_T u)(t) = u(t-2T) \) is M. A. Krasnoselskii operator (cf. [14]).

**Remark 1.2.** The operator \( K_T \) is well defined, because the initial function is defined on the interval \([-T, T]\). We notice that \( K_T \) maps \( M_S \) into itself. Indeed, consider the set \( C^1[-T, T] \) consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval \( (t_k, t_{k+1}) \). Introduce the set \( M_S^{(0)} = \{ u(\cdot) \in C^1[-T, T] : u(t) = u_0(t), t \in [-T, T] \} \). Then, \( K_T \) assigns to every function \( u(\cdot) \in M_S \) the function \( \bar{u}(\cdot) \in M_S^{(0)} \), translated to the right on the interval \([T, \infty)\). So, the function \( (K_T u)(t) \) coincides with \( u_0(t) \) on \([t_0, t_0 + 2T]\). Besides \( t_k - 2T = t_s \), and then

\[
(K_T u)(t_k) = \begin{cases} 
    u(t_k - 2T) = u_0(t_s) = 0, & t_k \in [T, 3T], \\
    u(t_k - 2T) = u(t_n) = 0, & t \in (3T, \infty),
  \end{cases}
\]

(1.7)

that is, \( (K_T u)(\cdot) \in M_S \).

**2. Main Results**

**Lemma 2.1.** If \( E \leq U_0 \), problem (1.4) has a solution \( u(\cdot) \in M_{SLU} \) iff the operator \( B \) has a fixed point in \( M_{SLU} \), that is,

\[
u(t) = B(u)(t).
\]

(2.1)
Proof. Let \( u(\cdot) \in M_{SL} \) be a solution of (1.4). Then, integrating (1.4) on the interval \([t_k, t] \subset [t_k, t_{k+1}] \) \((k = 0, 1, 2 \ldots)\), we obtain \( u(t) - u(t_k) = \int_{t_k}^t U(u)(s)ds \Rightarrow u(t) = \int_{t_k}^t U(u)(s)ds \), and then,

\[
 u(t) = \int_{t_k}^t U(u)(s)ds \quad \text{implies} \quad 0 = u(t_{k+1}) = \int_{t_k}^{t_{k+1}} U(u)(s)ds \quad \text{implies} \quad \int_{t_k}^{t_{k+1}} U(u)(s)ds = 0. \tag{2.2}
\]

Therefore, \( u(t) \) satisfies

\[
 u(t) = \int_{t_k}^t U(u)(s)ds \iff u(t) = \int_{t_k}^t U(u)(s)ds - \left( \frac{t - t_k}{t_{k+1} - t_k} \right) \int_{t_k}^{t_{k+1}} U(u)(s)ds, \tag{2.3}
\]

that is, \( u(\cdot) \) is a fixed point of \( B \).

Conversely, let \( u(\cdot) \in M_{SL} \) be a solution of \( u = B(u) \); that is,

\[
 u(t) = \int_{t_k}^t U(u)(s)ds - \left( \frac{t - t_k}{t_{k+1} - t_k} \right) \int_{t_k}^{t_{k+1}} U(u)(s)ds. \tag{2.4}
\]

Then, introducing \( \mu_0 = \mu T_0 \), we obtain

\[
 \left| \int_{t_k}^{t_{k+1}} U(u)(s)ds \right| \\ \leq \frac{2E}{C_0(Z_0 + R_0)} \int_{t_k}^{t_{k+1}} e^{\mu(t-t_k)} dt + \frac{1}{C_0 Z_0} \int_{t_k}^{t_{k+1}} |u(t)| dt \\ + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| \int_{t_k}^{t_{k+1}} |u(t)|^n dt + \kappa \frac{Z_0}{C_0} \int_{t_k}^{t_{k+1}} |u(t - 2T)| dt \\ + \kappa \frac{p}{C_0} \sum_{n=1}^{p} |r_n| \int_{t_k}^{t_{k+1}} |u(t - 2T)| dt + \kappa \left| \int_{t_k}^{t_{k+1}} u(t - 2T) dt \right| \\ \leq \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu(t_{k+1} - t_k)} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t_{k+1} - t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U^n_0 e^{\mu(T_0 - t_k)} dt \\ + \frac{\kappa U_0 e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t_{k+1} - t_k)} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| U^n_0 e^{-2\mu T} \\ \times \int_{t_k}^{t_{k+1}} e^{\mu(t-t_k)} dt + \kappa |u(t_{k+1} - 2T) - u(t_k - 2T)| \\ \leq \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu T_0} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U^n_0 e^{\mu T_0} - 1 \\ + \frac{U_0 \kappa e^{-2\mu T}}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| U^n_0 e^{-2\mu T} e^{\mu T_0} - 1
\[
\leq \frac{e^{\mu_0} - 1}{\mu C_0} \left( \frac{2U_0 e^{-\mu T}}{Z_0 + R_0} + \frac{U_0 (1 + \kappa e^{-2\mu T})}{Z_0} \right) + \frac{1}{\mu C_0} \sum_{n=1}^{p} |r_n| \left| U_0^n (1 + \kappa e^{-2\mu T})(e^{\mu_0} - 1) \right| \frac{1}{n} \\
\equiv M(\mu) .
\]

(2.5)

Let us assume that \( \int_{t_k}^{t_{k+1}} U(u)(t) dt = \beta > 0 \). We have just obtained that \( \beta \leq M(\mu) \). Then, for sufficiently large \( \mu > 0 \) (and sufficiently small \( T_0 > 0 \), one can reach the inequality \( M(\mu) < \beta \). Consequently, \( \int_{t_k}^{t_{k+1}} U(u)(t) dt = 0 \). It follows that \( u(t) = \int_t^{t_k} U(u)(s) ds \) and, after a differentiation, we obtain (1.4).

Lemma 2.1 is thus proved.

\[ \square \]

**Theorem 2.2.** Let \( S_T = \{ \tau_k \}_{k=0}^n, n \in N \) be the set of zeros of the initial function; that is, \( \nu_0(\tau_k) = 0 \) and \( \nu_0(\cdot) \in C[\tau_0, T] \). If \( E \leq U_0, |\nu_0(t)| \leq U_0 e^{\mu(t-\tau_0)}, t \in [\tau_k, \tau_{k+1}], \nu_0(0) = 0 \), then, there exists a unique oscillatory solution of the initial value problem (1.4), belonging to \( MSU \).

**Proof.** We show that \( B \) maps \( MSU \) into itself; that is, \( u \in MSU \Rightarrow B(u) \in MSU \).

Indeed, for every \( u(\cdot) \in MSU \), the function \( B(u)(t) \) is continuous on \( [t_0, \infty) \) and differentiable on every \( (t_k, t_{k+1}) \). We have also \( B(u)(t_k) = 0 \) and \( B(u)(t_{k+1}) = 0 \).

We show that \( |(Bu)(t)| \leq U_0 e^{\mu(t-t_k)}, t \in [t_k, t_{k+1}] \). (The last inequalities imply that \( B(u)(t) \) is bounded because \( e^{\mu(t-t_k)} \leq e^{\mu T_0}, t \in [T, \infty) \).)

We notice that \( |(t-t_k)/(t_{k+1}-t_k)| \leq 1, t \in [t_k, t_{k+1}] \). For sufficiently large \( \mu \), we obtain for \( t \in [t_k, t_{k+1}] \)

\[
|(Bu)(t)| \leq \left| \int_{t_k}^{t} U(u)(s) ds \right| + \left| \int_{t_k}^{t_{k+1}} U(u)(s) ds \right| \equiv B_1 + B_2 .
\]

(2.6)

We have

\[
B_1 \leq \left[ \frac{2}{C_0(Z_0 + R_0)} \int_{t_k}^{t} |E(s-T)| ds + \frac{1}{C_0 Z_0} \int_{t_k}^{t} |u(s)| ds + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| \int_{t_k}^{t} |u(s)| ds \right] + \frac{\kappa}{\mu C_0} \int_{t_k}^{t} |u(s-2T)| ds + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| \int_{t_k}^{t} |u(s-2T)| ds \right] + \kappa \int_{t_k}^{t} |u(s-2T)| ds
\]

\[
\leq \left[ \frac{2U_0 e^{-\mu T} e^{\mu(t-t_k)} - 1}{\mu} + \frac{U_0}{C_0 Z_0} e^{\mu(t-t_k)} - 1 \right] + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| \left| U_0^n \int_{t_k}^{t} e^{\mu(s-t_k)} ds \right|
\]

\[
+ \frac{\kappa U_0 e^{-2\mu T} e^{\mu(t-t_k)} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| \left| U_0^n e^{-2\mu T} \int_{t_k}^{t} e^{\mu(s-t_k)} ds \right| + \kappa |u(t-2T)|
\]

\[
\leq e^{\mu(t-t_k)} U_0 \left[ \frac{2e^{-\mu T} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U_0^n (1 + \kappa e^{-2\mu T})(e^{\mu_0} - 1)(1 + \kappa e^{-2\mu T})}{\mu Z_0 + R_0} \right]
\]

\[
+ \kappa e^{-2\mu T} ,
\]

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\[
B_2 \leq \left[ \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} e^{\mu(t_{k+1} - t_k)} - 1 \right] + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t_{k+1} - t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U_0^n e^{\mu(s-T)} ds_t \\
+ \frac{\kappa U_0 e^{-2\mu T} e^{\mu(t_{k+1} - t_k)} - 1}{Z_0 C_0} + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| U_0^n e^{-2\mu T} e^{\mu (s-T)} ds_t \\
+ \kappa |u(t_{k+1} - 2T) - u(t_k - 2T)| \left[ \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} e^{\mu T} - 1 \right] + \frac{U_0}{C_0 Z_0} \frac{e^{\mu T} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \\
+ \kappa \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| U_0^n e^{-2\mu T} e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \\
\leq e^{\mu(t-t_k)} U_0 \left( \frac{2e^{-\mu T} (e^{\mu T} - 1)}{Z_0 + R_0} + \frac{(e^{\mu T} - 1)(1 + \kappa e^{-2\mu T})}{Z_0} \\
+ \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \right) \frac{1 + \kappa e^{-2\mu T}}{\mu C_0} \\
+ e^{\mu(t-t_k)} U_0 \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T} (e^{\mu T} - 1)}{Z_0 + R_0} + \frac{(e^{\mu T} - 1)(1 + \kappa e^{-2\mu T})}{Z_0} \\
+ \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \right) \frac{1}{\mu C_0} \\
\leq e^{\mu(t-t_k)} U_0 \left[ \frac{2e^{-\mu T} e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0 + R_0} + \frac{e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0} \\
+ \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \right] \frac{1}{\mu C_0} \\
\leq e^{\mu(t-t_k)} U_0. \tag{2.7}
\]

Therefore, for sufficiently large \( \mu > 0 \), we obtain

\[
|(Bu)(t)| \leq e^{\mu(t-t_k)} U_0 \left[ \frac{2e^{-\mu T} e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0 + R_0} + \frac{e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0} \\
+ \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \right] \frac{1}{\mu C_0} \\
+ e^{\mu(t-t_k)} U_0 \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T} e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0 + R_0} + \frac{e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0} \\
+ \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \right) \frac{1}{\mu C_0} \\
\leq e^{\mu(t-t_k)} U_0 \left[ \frac{2e^{-\mu T} e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0 + R_0} + \frac{e^{\mu T} (1 + \kappa e^{-2\mu T})}{Z_0} \\
+ \sum_{n=1}^{p} |r_n| U_0^n e^{\mu T} - 1 \left(1 + \kappa e^{-2\mu T} \right) \right] \frac{1}{\mu C_0} \\
\leq e^{\mu(t-t_k)} U_0. \tag{2.8}
\]

Consequently, the operator \( B \) maps \( M_{SU} \) into itself.
We show that $B$ is a contractive operator. Indeed,

$$
|B(u)(t) - B(\bar{u})(t)| \leq \left| \int_{t_k}^{t} [U(u)(s) - U(\bar{u})(s)] ds \right| + \left| \int_{t_k}^{t_{k+1}} [U(u)(s) - U(\bar{u})(s)] ds \right|
$$

$$
\equiv B_1 + B_2, \quad t \in [t_k, t_{k+1}].
$$

We have

$$
B_1 \leq \left[ \frac{1}{C_0 Z_0} \int_{t_k}^{t} |u(s) - \bar{u}(s)| ds + \frac{1}{C_0} \sum_{n=1}^{p} n |r_n| \int_{t_k}^{t} |u^n(s) - \bar{u}^n(s)| ds \right.
$$

$$
+ \frac{\kappa}{Z_0 C_0} \int_{t_k}^{t} |u(s - 2T) - \bar{u}(s - 2T)| ds + \frac{\kappa}{C_0} \sum_{n=1}^{p} n |r_n| \int_{t_k}^{t} |u^n(s - 2T) - \bar{u}^n(s - 2T)| ds \left. \right]
$$

$$
+ \kappa \int_{t_k}^{t} \left( \dot{u}(s - 2T) - \dot{\bar{u}}(s - 2T) \right) ds
$$

$$
\leq \left[ \frac{\rho_k(u, \bar{u}) \rho^{(t_k)}}{C_0 Z_0} \frac{e^{\rho_k(u, \bar{u}) \rho^{(t_k)}} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} n |r_n| \text{ess sup} \left\{ |u^{n-1}(s) : s \in [t_k, t_{k+1}] \right\} \int_{t_k}^{t} |u(s) - \bar{u}(s)| ds \right.
$$

$$
+ \frac{\kappa}{Z_0 C_0} \rho_k(u, \bar{u}) e^{-2\mu T} \rho^{(t_k)} - 1 \mu
$$

$$
+ \frac{\kappa}{C_0} \sum_{n=1}^{p} n |r_n| \text{ess sup} \left\{ u^{n-1}(s - 2T) : s \in [t_k, t_{k+1}] \right\} \int_{t_k}^{t} |u(s - 2T) - \bar{u}(s - 2T)| ds \left. \right]
$$

$$
+ \kappa \rho_k \left( \dot{u}, \dot{\bar{u}} \right) e^{-2\mu T} \rho^{(t_k)} - 1 \mu
$$

$$
\leq e^{\rho^{(t_k)}} \left[ \frac{\rho_k(u, \bar{u})}{\mu C_0 Z_0} + \frac{\rho_k(u, \bar{u})}{C_0} \sum_{n=1}^{p} n |r_n| \|U_0^{n-1} e^{-2\mu T} e^{(n-1)\mu T} \|_{\mu C_0 Z_0} + \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{\mu Z_0 C_0} \right.
$$

$$
+ \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T} \rho^{(t_k)}}{\mu C_0 Z_0} \sum_{n=1}^{p} n |r_n| \|U_0^{n-1} e^{-2\mu T} e^{(n-1)\mu T} \|_{\mu C_0 Z_0} \left. \right]
$$

$$
+ e^{\rho^{(t_k)}} \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{\mu}
$$

$$
\leq e^{\rho^{(t_k)}} \rho_k \left( \dot{u}, \dot{\bar{u}} \right) \left[ \frac{1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^{p} n |r_n| \|U_0^{n-1} e^{-2\mu T} e^{(n-1)\mu T} \|_{\mu C_0 Z_0} + \frac{\kappa e^{-2\mu T}}{\mu} \right) \right]
$$

$$
\leq e^{\rho^{(t_k)}} \rho^{(t_k)}(u, \bar{u}) \left[ \frac{1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^{p} n |r_n| \|U_0^{n-1} e^{-2\mu T} e^{(n-1)\mu T} \|_{\mu C_0 Z_0} + \frac{\kappa e^{-2\mu T}}{\mu} \right) \right],
$$

$$
B_2 \leq \left[ \frac{1}{C_0 Z_0} \int_{t_k}^{t_{k+1}} |u(s) - \bar{u}(s)| ds + \frac{1}{C_0} \sum_{n=1}^{p} n |r_n| \int_{t_k}^{t_{k+1}} |u^n(s) - \bar{u}^n(s)| ds \right.
$$

$$
+ \frac{\kappa}{Z_0 C_0} \int_{t_k}^{t_{k+1}} |u(s - 2T) - \bar{u}(s - 2T)| ds + \frac{\kappa}{C_0} \sum_{n=1}^{p} n |r_n| \int_{t_k}^{t_{k+1}} |u^n(s - 2T) - \bar{u}^n(s - 2T)| ds \left. \right].
$$
Consequently,

\[
|B(u)(t) - B(\bar{u})(t)| \\
\leq e^{\mu t} \rho_{\mu}^{(k)}(u, \bar{u}) \left[ \frac{1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^{p} n|r_n|U_0^{n-1} e^{(n-1)\mu T} (1 + \kappa e^{-2\mu T}) \right) + \frac{\kappa e^{-2\mu T}}{\mu} \right] \\
\leq \rho_{\mu}^{(k)}(u, \bar{u}) \left[ \frac{e^{\mu T}}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^{p} n|r_n|U_0^{n-1} e^{(n-1)\mu T} (1 + \kappa e^{-2\mu T}) \right) + \frac{\kappa e^{-2\mu T}}{\mu} \right].
\]

(2.11)
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Therefore, \( \rho_k(Bu, B\bar{u}) \leq K_\rho \rho^{(k)}(u, \bar{u}). \)

It remains to estimate the derivative of \( B \).

We have

\[
|\dot{B}(u)(t) - \dot{B}(\bar{u})(t)| \leq |U(u)(s) - U(\bar{u})(s)|
+ \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| U(u)(s) - U(\bar{u})(s) \right| ds \right| \equiv \dot{B}_1 + \dot{B}_2. 
\]

We have

\[
\dot{B}_1 \leq \frac{1}{C_0 Z_0} |u(t) - \bar{u}(t)| + \frac{1}{C_0} \sum_{n=1}^{P} r_n |u^n(t) - \bar{u}^n(t)|
+ \frac{\kappa}{C_0 Z_0} |u(t - 2T) - \bar{u}(t - 2T)|
+ \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{C_0 Z_0}
+ \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{C_0 Z_0}.
\]

\[
\dot{B}_2 \leq \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| U(u)(s) - U(\bar{u})(s) \right| ds \leq \frac{1}{t_0} \int_{t_k}^{t_{k+1}} \left| U(u)(s) - U(\bar{u})(s) \right| ds
\leq \rho^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2 C_0 Z_0} \left( \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \frac{\sum_{n=1}^{P} r_n n U_0^{n-1} e^{(n-1)\mu(t_k - t_0)}}{\mu C_0 Z_0} \right)
+ \rho^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2 C_0 Z_0} \left( \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \frac{\sum_{n=1}^{P} r_n n U_0^{n-1} e^{(n-1)\mu(t_k - t_0)}}{\mu C_0 Z_0} \right).
\]

(2.13)
Finally, we summarize all inequalities needed for the applications:

3. Numerical Example

Therefore,

\[
|\dot{B}(u)(t) - \dot{B}(\bar{u})(t)| \leq e^{\mu(t - t_1)} \rho_{\mu}^{(k)}(u, \bar{u}) \left[\frac{1 + \kappa e^{-2\mu T}}{\mu C_0} + \frac{1}{Z_0} + \sum_{n=1}^{p} |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} + \kappa e^{-2\mu T}\right] \\
+ \rho_u^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2 C_0 l_0} \left[\frac{1 + \kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^{p} |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} \left(1 + \kappa e^{-2\mu T}\right)\right] \\
\leq \rho_{\mu}^{(k)}(u, \bar{u}) \left[\frac{e^{\mu T_0} - 1}{\mu^2 C_0 l_0} (1 + \kappa e^{-2\mu T}) \left(\frac{1}{Z_0} + \sum_{n=1}^{p} |r_n| n U_0^{n-1} e^{(n-1)\mu T_0}\right) + \kappa e^{-2\mu T}\right] \\
\equiv e^{\mu(t - t_1)} K_{U} \rho_{\mu}^{(k)}(u, \bar{u}).
\]

(2.14)

It follows \(\rho_k(\dot{B}(u), \dot{B}(\bar{u})) \leq e^{\mu(t - t_1)} K_{U} \rho_{\mu}^{(k)}(u, \bar{u}).\)

Then \(\rho_{\mu}^{(k)}(B(u), B(\bar{u})) \leq \max\{K_{U}, K_{UL}\} \rho_{\mu}^{(k)}(u, \bar{u}).\)

Consequently,

\[
\rho_{\mu}^{(k)}(Bu, B\bar{u}) \leq K \rho_{\mu}^{(k)}(u, \bar{u}) \quad (k = 0, 1, 2, \ldots),
\]

(2.15)

where \(K = \max\{K_{UL}, K_{UL}\} < 1\) does not depend on \(u\) and \(k\).

We have to verify that \(M_{SU}\) is \(j\)-bounded. Indeed, since \(j\) is an identity mapping,

\[
\rho_u^{(k)}(u, \bar{u}) \leq \rho_u^{(k)}(u, \bar{u}) < \infty \quad (n = 0, 1, 2, \ldots).
\]

(2.16)

Therefore, in view of the fixed point theorem for contractive mappings in uniform spaces (cf. [13]), the operator \(B\) has a unique fixed point, and it is an oscillatory solution of (1.4).

Theorem 2.2 is thus proved.

\[\square\]

3. Numerical Example

Finally, we summarize all inequalities needed for the applications:

\[
K_{UL} = \frac{e^{\mu_0}}{\mu^2} \left(\frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| n U_0^{n-1} e^{(n-1)\mu_0} \left(1 + \kappa e^{-2\mu T}\right)\right) + \frac{\kappa e^{-2\mu T}}{\mu} < 1,
\]

(3.1)

\[
K_{U} = \frac{e^{\mu_0} + \mu T_0 - 1}{\mu^2 C_0 l_0} \left(\frac{1}{Z_0} + \sum_{n=1}^{p} |r_n| n U_0^{n-1} e^{(n-1)\mu_0}\right) + \kappa e^{-2\mu T} < 1.
\]
Consider a line with the following specific parameters:

\[ \Lambda = 1 \text{m}, \quad L = 0.2 \mu \text{H/m}, \quad C = 80 \text{pF/m}, \]

\[ \nu = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \cdot 10^{-6} \cdot 80 \cdot 10^{-12}}} = \frac{1}{4 \cdot 10^{-9}} = 2.5 \cdot 10^8, \]

\[ Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.2 \cdot 10^{-6}}{80 \cdot 10^{-12}}} = 50 \Omega, \quad R_0 = 45 \Omega, \quad C_0 = 8 \text{pF} = 8 \cdot 10^{-12} \text{F}. \]

Then, \( T = \Lambda \sqrt{LC} = 4.10^{-9} \text{s}; \kappa = (Z_0 - R_0)/(Z_0 + R_0) = 1/19 = 0.0526. \) Let us check the propagation of millimeter waves \( \nu_0 = 10^{-3} \text{m}. \) We have

\[ f_0 = \frac{1}{\nu_0 \sqrt{LC}} = \frac{1}{10^{-3} \cdot 4 \cdot 10^{-9}} = 2.5 \cdot 10^{11} \text{Hz} \]

\[ \Rightarrow T_0 = \frac{1}{f_0} = \frac{1}{2.5 \cdot 10^{11}} = 4 \cdot 10^{-12} \text{sec}; \quad l = 2 \cdot 10^{-12} \text{sec}. \]

If we choose \( \mu = (1/4)10^{12}, \) then \( \mu T_0 = \mu_0 = 1, \mu \tau_0 = (1/2), \) and \( T = 4 \cdot 10^{-9} \cdot (1/4) \cdot 10^{12} T_0 = 1000 \cdot T_0. \)

Consequently, \( \mu T = (1/4)10^{12} \cdot 2 \cdot 10^{-8} = (1/2)10^4, \mu C_0 = (1/4)10^{12} \cdot 8 \cdot 10^410^{-12} = 2, \)

and \( \mu^2 C_0 = (1/2) \cdot 10^{-12}. \)

Since \( e^{\mu T} = e^{-3000} = 0, \) then the above inequalities (omitting the second one) become

\[ \frac{e}{100} + \sum_{n=1}^{p} |r_n| U_0^{n-1} e^n + e^{n-1} - 2 \cdot 2n \leq 1, \]

\[ K_{\mu} = 2 \left( e - \frac{1}{2} \right) \left( \frac{1}{50} + \sum_{n=1}^{p} |r_n| n U_0^{n-1} e^n \right) < 1. \]

If the \( V-I \) characteristic of the nonlinear resistive element is \( f(u) = -0, 12u + 0, 8u^3, \)

then \( U_0 \leq 0.41; \quad K_{\mu} = U_0 < 0.06. \) It follows that \( U_0 < 0.06. \)

References


