Research Article

Differential Subordination and Superordination for Srivastava-Attiya Operator

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Due to the well-known Srivastava-Attiya operator, we investigate here some results relating the p-valent of the operator with differential subordination and subordination. Further, we obtain some interesting results on sandwich-type theorem for the same.

1. Introduction and Motivation

Let \( \mathcal{H}(U) \) be the class of analytic functions in the open unit disc \( U \) and let \( \mathcal{H}[a,n] \) be the subclass of \( \mathcal{H}(U) \) consisting functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \), with \( a \in \mathcal{H}_0 = \mathcal{H}[0,1] \) and \( \mathcal{H} = \mathcal{H}[1,1] \). For two functions \( f_1 \) and \( f_2 \) analytic in \( U \), the function \( f_1 \) is subordinate to \( f_2 \), or \( f_2 \) superordinate to \( f_1 \), written as \( f_1 \prec f_2 \), if there exists a function \( w(z) \), analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f_1(z) = f_2(w(z)) \). In particular, if the function \( f_2 \) is univalent in \( U \), then \( f_1 < f_2 \) is equivalent to \( f_1(0) = f_2(0) \) and \( f_1(U) \subset f_2(U) \).

Let \( f, h \in \mathcal{H}(U) \) and \( \psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C} \). If \( f \) and \( \psi(f(z), zf'(z), z^2 f''(z); z) \) are univalent and \( f \) satisfies the second-order differential subordination

\[
\psi(f(z), zf'(z), z^2 f''(z); z) < h(z),
\]

then \( f \) is called a solution of the differential subordination. The univalent function \( F \) is called a dominant if \( f < F \) for all \( f \) satisfying (1.1). Miller and Mocanu discussed many interesting results containing the above mentioned subordination and also many applications of the field of differential subordination in [1]. In that direction, many differential subordination
and differential superordination problems for analytic functions defined by means of linear operators were investigated. See [2–11] for related results.

Let \( \mathcal{A}_p \) denote the class of functions of the form

\[
f(z) = z^n + \sum_{n=1}^{\infty} a_{n+p}z^{n+p} \quad (z \in \mathbb{U}, \ p \in \mathbb{N} = 1, 2, 3, \ldots),
\]

which are analytic and \( p \)-valent in \( \mathbb{U} \). For \( f \) satisfying (1.2), let the generalized Srivastava-Attiya operator [12] be denoted by

\[
J_{s,b} f(z) = G_{s,b} \ast f(z) \quad \left(b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots\right),
\]

where

\[
G_{s,b} = (1 + b)^s \left[ \varphi(z,s,b) - b^{-s} \right],
\]

with

\[
\varphi(z,s,b) = \frac{1}{b^s} + \frac{z^p}{(1 + b)^s} + \frac{z^{1+p}}{(2 + b)^s} + \cdots,
\]

and the symbol \( \ast \) denotes the usual Hadamard product (or convolution). From the equations, we can see that

\[
J_{s,b} f(z) = z^p + \sum_{n=1}^{\infty} \left( \frac{1 + b}{n + 1 + b} \right)^s a_{n+p}z^{n+p}.
\]

Note that for \( p = 1 \) in (1.6), \( J_{s,b} f(z) \) coincides with the Srivastava-Attiya operator [13]. Further, observe that for proper choices of \( s \) and \( b \), the operator \( J_{s,b} f(z) \) coincides with the following:

(i) \( J_{0,b} f(z) = f(z) \),
(ii) \( J_{1,b} f(z) = A(f)(z) \) [14],
(iii) \( J_{1,\gamma} f(z) = \partial_{\gamma}(f)(z), \ (\gamma > -1) \) [15, 16],
(iv) \( J_{\sigma,1} f(z) = I^{\sigma}(f)(z), \ (\sigma > 0) \) [17],
(v) \( J_{\alpha,\beta} f(z) = P_{\alpha,\beta}^{\sigma}(f)(z), \ (\alpha \geq 1, \ \beta > 1) \) [18].

Since the above mentioned operator, the generalized Srivastava-Attiya operator, \( J_{s,b} f(z) \) reduces to the well-known operators introduced and studied in the literature by suitably specializing the values of \( s \) and \( b \) and also in view of the several interesting properties and characteristics of well-known differential subordination results, we aim to associate these two motivating findings and obtain certain other related results. Further, we consider the differential superordination problems associated with the same operator. In addition, we also obtain interesting sandwich-type theorems.
The following definitions and theorems were discussed and will be needed to prove our results.

**Definition 1.1** (see [1], Definition 2.2b, page 21). Denote by $Q$ the set of all functions $q$ that are analytic and injective on $\mathbb{U} \setminus E(q)$ where

$$E(q) = \left\{ \zeta \in \partial \mathbb{U} : \lim_{z \to \zeta} q = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial \mathbb{U} \setminus E(q)$. Further let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$, and $Q(1) \equiv Q_1$.

**Definition 1.2** (see [1], Definition 2.3a, page 27). Let $\Omega$ be a set in $\mathbb{C}$, $q \in Q$, and let $n$ be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\varphi(c, d, e; z) \notin \Omega$ whenever $c = q(z)$, $d = k\zeta q'(\zeta)$, and

$$\text{Re}\left\{ \frac{e}{d} + 1 \right\} \geq k \text{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in \mathbb{U}$, $\zeta \in \partial \mathbb{U} \setminus E(q)$, and $k \geq n$. Let $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

**Definition 1.3** (see [19], Definition 3, page 817). Let $\Omega$ be a set in $\mathbb{C}$, $q \in \mathcal{E}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\varphi(c, d, e; \zeta) \notin \Omega$ whenever $c = q(z)$, $d = zq'(z)/m$, and

$$\text{Re}\left\{ \frac{e}{d} + 1 \right\} \geq \frac{1}{m} \text{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in \mathbb{U}$, $\zeta \in \partial \mathbb{U}$, and $m \geq n \geq 1$. Let $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

**Theorem 1.4** (see [1], Theorem 2.3b, page 28). Let $\varphi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $j(z) \in \mathcal{E}[a, n]$ satisfies

$$\varphi\left( j(z), zj'(z), z^2j''(z); z \right) \in \Omega,$$

then $j(z) < q(z)$.

**Theorem 1.5** (see [19], Theorem 1, page 818). Let $\varphi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $j \in Q(a)$ and $\varphi(j(z), zj'(z), z^2j''(z); z)$ is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{ \varphi\left( j(z), zj'(z), z^2j''(z); z \right) : z \in \mathbb{U} \right\}$$

implies $q(z) < j(z)$. 
2. Subordination Results Associated with Generalized Srivastava-Attiya Operator

Definition 2.1. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in Q_0 \cap \mathcal{A}[0,p] \). The class of admissible functions \( \Phi_f[\Omega, q] \) consists of those functions \( \phi : \mathbb{C} \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[
\phi(u, v, w; z) \not\in \Omega \tag{2.1}
\]

whenever

\[
u = q(\zeta), \quad v = \frac{kq(\zeta) - [p - (1 + b)]q(\zeta)}{1 + b} \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}) e^{\zeta} \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}), \]

\[
\Re \left\{ \frac{(1 + b)^2 v - [p - (1 + b)]^2 u}{(1 + b)v + [p - (1 + b)]u} + 2[p - (1 + b)] \right\} \geq k \Re \left\{ \frac{\bar{q}(\zeta)}{q(\zeta)} + 1 \right\},
\]

\( z \in U, \ z \in \partial U \setminus E(q), \) and \( k \geq p. \)

Theorem 2.2. Let \( \phi \in \Phi_f[\Omega, q] \). If \( f \in \mathcal{A}_p \) satisfies

\[
\{ \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) : z \in U \} \subset \Omega,
\]

then

\[
J_{s+2,b}f(z) \prec q(z) \quad (z \in U). \tag{2.4}
\]

Proof. The following relation obtained in [13]

\[
zJ'_{s+1,b}f(z) = [p - (1 + b)]J_{s+1,b}f(z) + (1 + b)J_{s,b}f(z)
\]

is equivalent to

\[
J_{s,b}f(z) = \frac{zJ'_{s+1,b}f(z) - [p - (1 + b)]J_{s+1,b}f(z)}{1 + b}
\]

and hence

\[
J_{s+1,b}f(z) = \frac{zJ'_{s+2,b}f(z) - [p - (1 + b)]J_{s+2,b}f(z)}{1 + b}.
\]

Define the analytic function \( j \) in \( U \) by

\[
j(z) = J_{s+2,b}f(z), \tag{2.8}
\]
and then we get

\[
J_{s+1,b}f(z) = \frac{zj'(z) - [p - (1 + b)]j(z)}{1 + b},
\]

\[
J_{s,b}f(z) = \frac{z^2j'(z) + (1 - 2[p - (1 + b)])zj'(z) + [p - (1 + b)]^2j(z)}{(1 + b)^2}.
\]

Further, let us define the transformations from \( \mathbb{C}^3 \) to \( \mathbb{C} \) by

\[
u = c, \quad \psi = \frac{d \cdot c}{1 + b},
\]

\[
w = \frac{e + (1 - 2[p - (1 + b)])d + [p - (1 + b)]^2c}{(1 + b)^2}.
\]

Let

\[
\psi(c, d, e; z) = \phi(u, v, w; z),
\]

\[
\phi(u, v, w; z) = \phi\left(c, \frac{d - [p - (1 + b)]c}{1 + b}, \frac{e + (1 - 2[p - (1 + b)])d + [p - (1 + b)]^2c}{(1 + b)^2}; z\right).
\]

The proof will make use of Theorem 1.4. Using (2.8) and (2.9), from (2.12) we obtain

\[
\psi\left(j(z), zj'(z), z^2j''(z); z\right) = \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z).
\]

Hence (2.3) becomes

\[
\psi\left(j(z), zj'(z), z^2j''(z); z\right) \in \Omega.
\]

Note that

\[
\frac{e}{d} + 1 = \frac{(1 + b)^2w - [p - (1 + b)]^2u}{(1 + b)v + [p - (1 + b)]u + 2[p - (1 + b)]},
\]

and since the admissibility condition for \( \phi \in \Phi_I[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.2, hence \( \psi \in \Psi_p[\Omega, q] \), and by Theorem 1.4,

\[
j(z) < q(z),
\]
In the case $\phi(u, v, w; z) = v$, we have the following example.

Example 2.3. Let the class of admissible functions $\Phi_{f_\nu} [\Omega, q]$ consist of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$v = \frac{k \zeta q' (\zeta) - [p - (1 + b)] q(\zeta)}{1 + b} \not\in \Omega,$$

(2.18)

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq p$ and $\phi \in \Phi_{f_\nu} [\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies

$$J_{s+1, b} f(z) \subset \Omega,$$

(2.19)

then

$$J_{s+2, b} f(z) < q(z) \quad (z \in U).$$

(2.20)

If $\Omega \not\subset \mathbb{C}$ is a simply connected domain, then $\Omega \in h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$ and the class is written as $\Phi_{f_\nu} [h, q]$. The following result follows immediately from Theorem 2.2.

Theorem 2.4. Let $\phi \in \Phi_{f_\nu} [\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies

$$\phi (J_{s+2, b} f(z), J_{s+1, b} f(z), J_{s, b} f(z); z) < h(z),$$

(2.21)

then

$$J_{s+2, b} f(z) < q(z).$$

(2.22)

The next result occurs when the behavior of $q$ on $\partial U$ is not known.

Corollary 2.5. Let $\Omega \subset \mathbb{C}$, $q$ be univalent in $U$ and $q(0) = 0$. Let $\phi \in \Phi_{f_\nu} [\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}_p$ and

$$\phi (J_{s+2, b} f(z), J_{s+1, b} f(z), J_{s, b} f(z); z) \in \Omega,$$

(2.23)

then

$$J_{s+2, b} f(z) < q(z).$$

(2.24)
Theorem 2.7. Let $q$ and $q$ have a solution $f$ with $q(0) = 0$ and set $q(z) = q(\rho z)$ and $h(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfy one of the following conditions:

1. $\phi \in \Phi[1, h, q]$, for some $\rho \in (0, 1)$, or
2. there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi[1, h, q]$, for all $\rho_0 \in (0, 1)$.

If $f \in \mathcal{A}_p$ satisfies (2.21), then

$$J_{s+2, h} f(z) < q(z).$$

Proof. From Theorem 2.2, we see that $J_{s+2, h} f(z) < q(z)$ and the proof is complete. \hfill \Box

Theorem 2.6. Let $h$ and $q$ be univalent in $\mathcal{U}$, with $q(0) = 0$ and set $q(z) = q(\rho z)$ and $h(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfy one of the following conditions:

1. $\phi \in \Phi[1, h, q]$, for some $\rho \in (0, 1)$, or
2. there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi[1, h, q]$, for all $\rho_0 \in (0, 1)$.

If $f \in \mathcal{A}_p$ satisfies (2.21), then

$$J_{s+2, h} f(z) < q(z).$$

Proof. The proof is similar to the one in [1] and therefore is omitted. \hfill \Box

The next results give the best dominant of the differential subordination (2.21).

Theorem 2.7. Let $h$ be univalent in $\mathcal{U}$. Let $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. Suppose that the differential equation

$$\phi \left(q(z), zq'(z), z^2q''(z); z\right) = h(z)$$

has a solution $q$ with $q(0) = 0$ and satisfy one of the following conditions:

1. $q \in \mathcal{Q}_0$ and $\phi \in \Phi[1, h, q]$,
2. $q$ is univalent in $\mathcal{U}$ and $\phi \in \Phi[1, h, q]$, for some $\rho \in (0, 1)$, or
3. $q$ is univalent in $\mathcal{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi[1, h, q]$, for all $\rho_0 \in (0, 1)$.

If $f \in \mathcal{A}_p$ satisfies (2.21), then

$$J_{s+2, h} f(z) < q(z),$$

and $q$ is the best dominant.

Proof. Following the same arguments in [1], we deduce that $q$ is a dominant from Theorem 2.4 and Theorem 2.6. Since $q$ satisfies (2.26), it is also a solution of (2.21) and therefore $q$ will be dominated by all dominants. Hence $q$ is the best dominant. \hfill \Box

Definition 2.8. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{A}_0$. The class of admissible functions $\Phi[1, \mathcal{U}, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) - b q(\zeta)}{1 + b} \quad \left( b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N} \right),$$

$$\text{Re} \left\{ \frac{(1 + b)^2 w - b^2 u}{(1 + b) v + bu} - 2b \right\} \geq k \text{Re} \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$$z \in \mathcal{U}, \quad \zeta \in \partial \mathcal{U} \setminus E(q), \text{ and } k \geq 1.$$
**Theorem 2.9.** Let \( \phi \in \Phi_{f_1, \mathcal{A}_p}[\Omega, q] \). If \( f \in \mathcal{A}_p \) satisfies
\[
\left\{ \phi \left( \frac{J_{s+1,k}f(z)}{z^{p-1}}, \frac{J_{s+1,k}f(z)}{z^{p-1}}, \frac{J_{s,k}f(z)}{z^{p-1}}; z \right) : z \in U \right\} \subset \Omega,
\]
then
\[
\frac{J_{s+2,k}f(z)}{z^{p-1}} < q(z) \quad (z \in U).
\]

**Proof.** Define the analytic function \( j \) in \( U \) by
\[
j(z) = \frac{J_{s+2,k}f(z)}{z^{p-1}}.
\]
Using the relations (2.5) and (2.32), we get
\[
\frac{J_{s+1,k}f(z)}{z^{p-1}} = \frac{zj'(z) - bj(z)}{1 + b}, \\
\frac{J_{s,k}f(z)}{z^{p-1}} = \frac{z^2j''(z) + (2b + 1)zj'(z) + b^2j(z)}{(1 + b)^2}.
\]
Further, let us define the transformations from \( \mathbb{C}^3 \) to \( \mathbb{C} \) by
\[
u = \frac{d + bc}{1 + b}, \\
w = \frac{e + (2b + 1)d + b^2c}{(1 + b)^2}.
\]
Let
\[
\psi(c, d, e; z) = \phi(u, v, w; z) = \phi \left( c, \frac{d + bc}{1 + b}, \frac{e + (2b + 1)d + b^2c}{(1 + b)^2}; z \right).
\]
The proof will make use of Theorem 1.4. Using (2.32) and (2.33), from (2.35) we obtain
\[
\psi \left( j(z), zj'(z), z^2j''(z); z \right) = \phi \left( \frac{J_{s+2,k}f(z)}{z^{p-1}}, \frac{J_{s+1,k}f(z)}{z^{p-1}}, \frac{J_{s,k}f(z)}{z^{p-1}}; z \right).
\]
Hence (2.30) becomes
\[
\psi \left( j(z), zj'(z), z^2j''(z); z \right) \in \Omega.
\]
Note that

\[
\frac{e}{d} + 1 = \frac{(1 + b)^2 w - b^2 u}{d} - 2b, 
\]

and since the admissibility condition for \( \phi \in \Phi_{J,1}[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.2, hence \( \psi \in \Psi[\Omega, q] \), and by Theorem 1.4,

\[
j(z) < q(z),
\]

or

\[
\frac{J_{s+2,b}f(z)}{z^{p-1}} < q(z).
\]

In the case \( \phi(u, v, w; z) = v - u \), we have the following example.

**Example 2.10.** Let the class of admissible functions \( \Phi_{J,1}[\Omega, q] \) consist of those functions \( \phi : \mathbb{C}^3 \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[
v - u = \frac{k_1 q(\zeta) - pq(\zeta)}{1 + b} \notin \Omega,
\]

\( z \in U, \ \zeta \in \partial U \setminus E(q) \), and \( k \geq p \) and \( \phi \in \Phi_{J,1}[\Omega, q] \). If \( f \in A_p \) satisfies

\[
\frac{J_{s+2,b}f(z)}{z^{p-1}} < q(z), \quad (z \in U),
\]

then

\[
\frac{J_{s+2,b}f(z)}{z^{p-1}} < q(z), \quad (z \in U).
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega \in h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \) and the class is written as \( \Phi_{J,1}[h, q] \). The following result follows immediately from Theorem 2.9.

**Theorem 2.11.** Let \( \phi \in \Phi_{J,1}[\Omega, q] \). If \( f \in A_p \) satisfies

\[
\phi\left( \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}} ; z \right) < h(z),
\]

then

\[
\frac{J_{s+2,b}f(z)}{z^{p-1}} < q(z).
\]
Definition 2.12. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{A}$. The class of admissible functions $\Phi_{1,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = q(\zeta) + \frac{k\zeta q'(\zeta)}{(1 + b)q(\zeta)} \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}),$$

$$\Re \left\{ \frac{(w - u)(1 + b)u}{v - u} + (1 + b)(w - 3u) \right\} \geq k \Re \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$$z \in U, \; \zeta \in \partial U \setminus E(q), \text{ and } k \geq 1.$$

Theorem 2.13. Let $\phi \in \Phi_{1,2}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies

$$\left\{ \phi \left( \frac{J_{s+2,b,f}(z)}{J_{s+3,b,f}(z)}, \frac{J_{s+1,b,f}(z)}{J_{s+2,b,f}(z)}, \frac{J_{s,b,f}(z)}{J_{s+1,b,f}(z)} ; z \in U \right) \right\} \subset \Omega,$$

then

$$\frac{J_{s+2,b,f}(z)}{J_{s+3,b,f}(z)} < q(z) \quad (z \in U).$$

Proof. Define the analytic function $j$ in $U$ by

$$j(z) = \frac{J_{s+2,b,f}(z)}{J_{s+3,b,f}(z)}.$$

Differentiating (2.50) yields

$$\frac{zj'(z)}{j(z)} = \frac{zJ'_{s+2,b,f}(z)}{J_{s+2,b,f}(z)} - \frac{J'_{s+3,b,f}(z)}{J_{s+3,b,f}(z)}.$$

From the relation (2.5) we get

$$\frac{zJ'_{s+2,b,f}(z)}{J_{s+2,b,f}(z)} = [p - (1 + b)] + (1 + b)j + \frac{zj'(z)}{j(z)},$$

and hence

$$\frac{J_{s+1,b,f}(z)}{J_{s+2,b,f}(z)} = j(z) + \frac{zj'(z)}{(1 + b)j(z)}.$$
Further computations show that
\[
\frac{J_{s,k}f(z)}{J_{s+1,k}f(z)} = j(z) + \frac{[2(1+b)j(z) + 1]zj'(z) + z^2j''(z)}{(1+b)^2j(z)^2 + (1+b)zj'(z)}. \tag{2.54}
\]

Let us define the transformations from \( \mathbb{C}^3 \) to \( \mathbb{C} \) by
\[
u = c + \frac{d}{(1+b)c}, \quad \psi = c + \frac{[2(b+1)c + 1]d + e}{(1+b)^2c^2 + (1+b)d}. \tag{2.55}\]

Let
\[
\varphi(c, d, e; z) = \phi(u, v, w; z) = \phi\left( c, c + \frac{d}{(1+b)c}, c + \frac{[2(b+1)c + 1]d + e}{(1+b)^2c^2 + (1+b)d}; z \right). \tag{2.56}
\]

The proof will make use of Theorem 1.4. Using (2.50), (2.53) and (2.54), from (2.56) we obtain
\[
\varphi\left( j(z), j'(z), z^2j''(z); z \right) = \phi\left( J_{s+2,k}f(z), J_{s+1,k}f(z), \frac{J_{s,k}f(z)}{J_{s+1,k}f(z)}; z \right). \tag{2.57}
\]

Hence (2.48) becomes
\[
\varphi\left( j(z), j'(z), z^2j''(z); z \right) \in \Omega. \tag{2.58}
\]

Note that
\[
\frac{e}{d} + 1 = \frac{(w-u)(1+b)u}{v-u} + (1+b)(w-3u), \tag{2.59}
\]

and since the admissibility condition for \( \phi \in \Phi_{1,2}[\Omega, q] \) is equivalent to the admissibility condition for \( \varphi \) as given in Definition 1.2, hence \( \varphi \in \Psi[\Omega, q] \) and by Theorem 1.4,
\[
\psi < q(z), \tag{2.60}
\]

or
\[
\frac{J_{s+2,k}f(z)}{J_{s+3,k}f(z)} < q(z). \tag{2.61}
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega \in h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \) and the class is written as \( \Phi_{1,2}[h, q] \). The following result follows immediately from Theorem 2.13.
Theorem 2.14. Let $\phi \in \Phi_{J,2}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies
\[
\phi \left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}; z \right) < h(z),
\]
then
\[
\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \times q(z).
\]

3. Superordination Results Associated with Generalized Srivastava-Attiya Operator

Definition 3.1. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{A}[0, p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_j[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \partial U \to \mathbb{C}$ that satisfy the admissibility condition:
\[
\phi(u, v, w; \zeta) \notin \Omega
\]
whenever
\[
u = q(z), \quad v = \frac{zq'(z) - m[p - (1 + b)]q(z)}{m(1 + b)} \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}),
\]
\[
\text{Re} \left\{ \frac{(1 + b)^2 \omega - [p - (1 + b)]^2 u}{(1 + b)v + [p - (1 + b)]u} + 2[p - (1 + b)] \right\} \geq \frac{1}{m} \text{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},
\]
\[
z \in U, \quad \zeta \in \partial U, \text{ and } m \geq p.
\]

Theorem 3.2. Let $\phi \in \Phi_j[\Omega, q]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f \in \mathcal{Q}_0$ and
\[
\phi \left( J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z \right)
\]
is univalent in $U$, then
\[
\Omega \subset \{ \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) : z \in U \}
\]
implies that
\[
q(z) < J_{s+2,b}f(z).
\]

Proof. From (2.13) and (3.4), we have
\[
\Omega \subset \{ \psi \left( j(z), zj'(z), z^2j''(z); z \right) : z \in U \}.
\]
From (2.10), we see that the admissibility condition for \( \phi \in \Phi'_f[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.3. Hence \( \psi \in \Psi'[\Omega, q] \), and by Theorem 1.5, \( q(z) < j(z) \) or

\[
q(z) < J_{s+2,b}f(z). \tag{3.7}
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega \in h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \) and the class is written as \( \Phi'_f[h, q] \). The next result follows immediately from Theorem 3.2.

**Theorem 3.3.** Let \( h \) be analytic in \( U \) and \( \phi \in \Phi'_f[\Omega, q] \). If \( f \in \mathcal{A}_p, J_{s+2,b}f(z) \in Q_0 \) and

\[
\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z)
\]

is univalent in \( U \), then

\[
h(z) < \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z), \tag{3.9}
\]

and then

\[
q(z) < J_{s+2,b}f(z). \tag{3.10}
\]

Theorems 3.2 and 3.3 can only be used to obtain subordinants for differential superordination of the form (3.4) and (3.9). The following theorems prove the existence of the best subordinant of (3.9) for certain \( \phi \).

**Theorem 3.4.** Let \( h \) be analytic in \( U \) and \( \phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C} \). Suppose that the differential equation

\[
\phi \left( q(z), zq'(z), z^2q''(z); z \right) = h(z) \tag{3.11}
\]

has a solution \( q \in Q_0 \). If \( \phi \in \Phi'_f[\Omega, q] \), \( f \in \mathcal{A}_p, J_{s+2,b}f(z) \in Q_0 \), and

\[
\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z)
\]

is univalent in \( U \), then

\[
h(z) < \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) \tag{3.13}
\]

implies that

\[
q(z) < J_{s+2,b}f(z), \tag{3.14}
\]

and \( q(z) \) is the best subordinant.
Corollary 3.5. Let $h_1$ and $q_1$ be analytic in $U$, and let $h_2$ be univalent function in $U$, $q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_1 [h_2, q_2] \cap \Phi_1 [h_1, q_1]$. If $f \in A_p$, $J_{s+2,b}f(z) \in A[0, p] \cap Q_0$, and

$$\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z)$$

(3.15)

is univalent in $U$, then

$$h_1(z) < \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) < h_2(z)$$

(3.16)

implies that

$$q_1(z) < J_{s+2,b}f(z) < q_2(z).$$

(3.17)

Definition 3.6. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in A_0$ with $zq(z) \neq 0$. The class of admissible functions $\Phi_{s,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C} \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega$$

(3.18)

whenever

$$u = q(z), \quad v = \frac{zq'\{z\} - mbq(z)}{m(1 + b)} \quad \left( b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N} \right),$$

$$\text{Re}\left\{ \frac{(1 + b)^2w - b^2u}{(1 + b)v + bu} - 2b \right\} \geq \frac{1}{m} \text{Re}\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

(3.19)

$$z \in U, \quad \zeta \in \partial U, \text{ and } m \geq 1.$$

The following result is associated with Theorem 2.9.

Theorem 3.7. Let $\phi \in \Phi_{s,1}[\Omega, q]$. If $f \in A_p$, $J_{s+2,b}f(z)/z^{p-1} \in Q_0$, and

$$\phi\left( \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z \right)$$

(3.20)

is univalent in $U$, then

$$\Omega \subset \left\{ \phi\left( \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z \right) : z \in U \right\}$$

(3.21)
implies that

\[ q(z) < \frac{J_{s+2,b}f(z)}{z^{p-1}}. \]  

(3.22)

**Proof.** From (3.26) and (3.21), we have

\[ \Omega \subset \{ \phi(j(z), zj'(z), z^2j''(z); z) : z \in U \}. \]  

(3.23)

From (2.34), we see that the admissibility condition for \( \phi \in \Phi'_{j,1}[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as in Definition 1.3. Hence \( \psi \in \Psi'[\Omega, q] \), and by Theorem 1.5, \( q(z) < j(z) \) or

\[ q(z) < \frac{J_{s+2,b}f(z)}{z^{p-1}}. \]  

(3.24)

\[ \square \]

If \( \Omega \neq \emptyset \) is a simply connected domain, then \( \Omega \in h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \) and the class is written as \( \Phi'_{j,1}[h, q] \). The next result follows immediately from Theorem 3.7.

**Theorem 3.8.** Let \( q \in \mathcal{H}_0 \), and let \( h \) be analytic on \( U \), and let \( \phi \in \Phi'_{j,1}[\Omega, q] \). If \( f \in \mathcal{A}_p \),

\[ J_{s+2,b}f(z)/z^{p-1} \in \mathcal{Q}_0, \]

and

\[ \phi \left( \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z \right) \]  

(3.25)

is univalent in \( U \), then

\[ h(z) < \phi \left( \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z \right) \]  

(3.26)

implies that

\[ q(z) < \frac{J_{s+2,b}f(z)}{z^{p-1}}. \]  

(3.27)

Combining Theorems 2.11 and 3.8, we obtain the following sandwich-type theorem.

**Corollary 3.9.** Let \( h_1 \) and \( q_1 \) be analytic in \( U \), let \( h_2 \) be univalent function in \( U \), \( q_2 \in \mathcal{Q}_0 \) with \( q_1(0) = q_2(0) = 0 \), and \( \phi \in \Phi_{j,1}[h_2, q_2] \cap \Phi'_{j,1}[h_1, q_1] \). If \( f \in \mathcal{A}_p \),

\[ J_{s+2,b}f(z)/z^{p-1} \in \mathcal{H}_0 \cap \mathcal{Q}_0, \]

and

\[ \phi \left( \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z \right) \]  

(3.28)
is univalent in \( U \), then

\[
\begin{align*}
    h_1(z) < \phi\left( \frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{sb}f(z)}{z^{p-1}} ; z \right) < h_2(z)
\end{align*}
\] (3.29)

implies that

\[
q_1(z) < \frac{J_{s+2,b}f(z)}{z^{p-1}} < q_2(z).
\] (3.30)

**Definition 3.10.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q(z) \not= 0 \), \( zq'(z) \not= 0 \), and \( q \in \mathcal{K} \). The class of admissible functions \( \Phi'_{j2}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) that satisfy the admissibility condition:

\[
\phi(u, v, w; \zeta) \not\in \Omega
\] (3.31)

whenever

\[
\begin{align*}
    u &= q(z), \quad v = q(z) + \frac{zq'(z)}{m(1+b)q(z)}, \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}), \\
    \text{Re}\left\{ \frac{(w-u)(1+b)u}{v-u} + (1+b)(w-3u) \right\} &\geq \frac{1}{m} \text{Re}\left\{ \frac{zq'(z)}{q(z)} + 1 \right\},
\end{align*}
\] (3.32)

\( z \in U, \ \zeta \in \partial U \), and \( m \geq 1 \).

The following result is associated with Theorem 2.13.

**Theorem 3.11.** Let \( \phi \in \Phi'_{j2}[\Omega, q] \). If \( f \in \mathcal{A}_p, J_{s+2,b}f(z)/J_{s+3,b}f(z) \in Q_1 \), and

\[
\phi\left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{sb}f(z)}{J_{s+1,b}f(z)} ; z \right)
\] (3.33)

is univalent in \( U \), then

\[
\Omega \subset \left\{ \phi\left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{sb}f(z)}{J_{s+1,b}f(z)} ; z \right) : z \in U \right\}
\] (3.34)

implies that

\[
q(z) < \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}.
\] (3.35)

**Proof.** From (2.57) and (3.34), we have

\[
\Omega \subset \left\{ \phi\left( j(z), j'(z), z^2j''(z) ; z \right) : z \in U \right\}.
\] (3.36)
Corollary 3.13. Let \( \Omega \) be univalent in \( U \) and let \( h \) be univalent function in \( U \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \) and the class is written as \( \Phi'_s[h,q] \). The next result follows immediately from Theorem 3.11 as in the previous section.

Theorem 3.12. Let \( q \in \mathcal{H} \), let \( h \) be analytic in \( U \), and let \( \phi \in \Phi'_s[h,q] \). If \( f \in \mathcal{A}_p \), \( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \in Q_1 \) and

\[
\phi\left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \right) \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} \neq \frac{q(z)}{f(z)}
\]

is univalent in \( U \), then

\[
h(z) < \phi\left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \right) \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} \neq \frac{q(z)}{f(z)}
\]

implies that

\[
q(z) < \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}.
\]

Combining Theorems 2.14 and 3.12, we obtain the following sandwich-type theorem.

Corollary 3.13. Let \( h_1 \) and \( q_1 \) be analytic in \( U \), let \( h_2 \) be univalent function in \( U \), \( q_2 \in Q_0 \) with \( q_1(0) = q_2(0) = 0 \), and \( \phi \in \Phi'_s[h_2,q_2] \cap \Phi'_s[h_1,q_1] \). If \( f \in \mathcal{A}_p \), \( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \in \mathcal{H} \cap Q_1 \), and

\[
\phi\left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \right) \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} \neq \frac{q(z)}{f(z)}
\]

is univalent in \( U \), then

\[
h_1(z) < \phi\left( \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \right) \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} \neq \frac{q(z)}{f(z)} < h_2(z)
\]

implies that

\[
q_1(z) < \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} < q(z).
\]
Other work related to certain operators concerning the subordination and superordination can be found in [20–25].

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References


