Research Article

Oscillation Criteria for Even Order Neutral Equations with Distributed Deviating Argument

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We present new oscillation criteria for the even order neutral delay differential equations with distributed deviating argument

\[ r(t)\varphi(x(t))Z^{(n-1)}(t) + \int_a^b p(t,\xi)f[x(g(t,\xi))]d\sigma(\xi) = 0, \quad t \geq t_0, \]

where \( Z(t) = x(t) + q(t)x(t-\tau) \), \( \tau \geq 0 \) and \( n \) is an even positive integer. We assume that

(A1) \( r, q \in C(I, R) \) and \( 0 \leq q(t) \leq 1, \ r(t) > 0 \) for \( t \in I, \int_0^\infty 1/r(s)ds = \infty, \ I = [t_0, \infty) \);

(A2) \( \varphi \in C^1([0, \infty)), \ \varphi(x) > 0 \) for \( x \neq 0 \);

(A3) \( f \in C([0, \infty)), \ xf(x) > 0 \) for \( x \neq 0 \);

(A4) \( p \in C(I \times [a, b], [0, \infty)) \) and \( p(t,\xi) \) is not eventually zero on any half linear \( [t_0, \infty) \times [a, b], t_0 \geq t_0; \)

(A5) \( g \in C(I \times [a, b], [0, \infty)), \ g(t,\xi) \leq t \) for \( \xi \in [a, b], \ g(t,\xi) \) has a continuous and positive partial derivative on \( I \times [a, b] \) with respect to \( t \) and nondecreasing with respect to \( \xi \), respectively, \( \lim \inf_{t \to \infty} g(t,\xi) = \infty \) for \( \xi \in [a, b] \);
\[ (A_n) \quad \sigma \in C([a,b],R) \text{ is nondecreasing, and the integral of (1.1) is in the sense of Riemann-Stieltjes.} \]

We restrict our attention to those solutions \( x(t) \) of (1.1) which exist on some half linear \([t_0, \infty)\) and satisfy \( \sup \{|x(t)| : t \geq t_0\} \neq 0 \) for any \( T \geq t_0 \). As usual, such a solution of (1.1) is called oscillatory if the set of its zeros is unbounded from above; otherwise, it is said to be nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The oscillatory behavior of solutions of higher-order neutral differential equations is of both theoretical and practical interest. There have been some results on the oscillatory and asymptotic behavior of even order neutral equations. We mention here [1–12]. The oscillation problem for nonlinear delay equation such as

\[
[r(t)x'(t)]' + q(t)f(x(\sigma(t))) = 0, \quad t > t_0 \tag{1.2}
\]

as well as for the the linear ordinary differential equation

\[
[r(t)x'(t)]' + p(t)x(t) + q(t)x(t) = 0, \quad t > t_0 \tag{1.3}
\]

and the neutral delay differential equation

\[
(x(t) + q(t)x(t-\sigma))'' + p(t)x(t-\tau) = 0 \tag{1.4}
\]

has been studied by many authors with different methods. In [13], Rogovchenko established some general oscillation criteria for second-order nonlinear differential equation:

\[
(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0. \tag{1.5}
\]

In [14], the authors discussed the following neutral equations of the form

\[
[x(t) + c(t)x(t-\tau)]^{(n)} + \int_a^b p(t,\xi)x[g(t,\xi)]d\sigma(\xi) = 0, \quad t \geq t_0 \tag{1.6}
\]

and obtained the following results.

**Theorem A** (see [14, Theorem 2]). Assume that there exist functions \( H(t, s) \in \mathcal{C}'(D; R), h(t, s) \in \mathcal{C}(D_0; R), \) and \( \rho(t) \in \mathcal{C}'([t_0, \infty),(0,0)), \) such that

(I) \( H(t, t) = 0, H(t, s) > 0; \)

(II) \( H'(t, s) \leq 0, \text{ and } -H'(t, s) - H(t, s)(\rho'(s)/\rho(s)) = h(t, s)\sqrt{H(t, s)}, \) and

\[
0 < \inf_{s \geq 0} \left[ \lim_{t \to \infty} \inf_{t \geq t_0} \frac{H(t, s)}{H(t, t_0)} \right] \leq \infty; \tag{C_1}
\]

\[
\lim_{t \to \infty} \sup_{t \geq t_0} \frac{1}{H(t, t_0)} \int_{t}^{t} \frac{\rho(s)h^2(t, s)}{g^{n-2}(s, a)g'(s, a)} ds < \infty. \tag{C_2}
\]
If there exists a function \( \varphi(t) \in C([t_0, \infty), \mathbb{R}) \) satisfying

\[
\lim_{t \to \infty} \sup_{t_0 < u < t} \frac{1}{H(t,u)} \int_{t_0}^{t} H(t,s) \rho(s) \left( p(s,\xi) \left( 1 - c[g(s,\xi)] \right) \right) d\sigma(\xi) \geq 0, \quad u \geq t_0,
\]

\[
\lim_{t \to \infty} \sup_{t_0 < u < t} \frac{1}{H(t,u)} \int_{t_0}^{t} g'(u,a) g^{n-2}(u,a) \varphi^2(u) du = \infty, \quad \varphi_+(u) = \max_{u \geq t_0} \{ \varphi(u,0) \},\]

then every solution of (1.6) is oscillatory.

We will use the function class \( W \) to study the oscillation criteria for (1.1). Let \( D = \{(t,s) \mid t \geq s \geq t_0 \} \), and \( D_0 = \{(t,s) \mid t > s \geq t_0 \} \). We say that a continuous function \( H(t,s) \in C'(D, \mathbb{R}) \) belongs to the class \( W \) if

\( (A_7) \) \( H(t,t) = 0 \) and \( H(t,s) > 0 \) for \( -\infty < s < t < +\infty \);

\( (A_8) \) \( H \) has a continuous partial derivative \( \partial H/\partial s \) satisfying, for some \( h \in L_{\text{loc}}(D, \mathbb{R}) \), the condition \( \partial H/\partial s = -h(t,s)\sqrt{H(t,s)} \).

The purpose of this paper is to further improve Theorem A by Wang et al. [14], using a generalized Riccati transformation and developing ideas exploited by the Rogovchenko and Tuncay [13], we establish some new oscillation criteria for (1.1), which remove condition \( (C_2) \) in Theorem A by Wang et al. [14]; this complements and extends the results in [14]. In addition, we will make use of the following conditions.

\( (S_1) \) There exists a positive real number \( M \) such that \( |f(\pm uv)| \geq M |f(u)f(v) | \) for \( uv > 0 \).

**Lemma 1.1.** If \( a > 0, b \geq 0 \), then

\[
-ax^2 + bx \leq -\frac{a}{2}x^2 + \frac{b^2}{2a}.
\]

**Lemma 1.2** (Kiguradze [15]). Let \( u(t) \) be a positive and \( n \) times differentiable function on \( \mathbb{R} \). If \( u^{(n)}(t) \) is of constant sign and identically zero on any ray \( (t_1, +\infty) \) for \( (t_1 > 0) \), then there exists a \( t_u \geq t_1 \) and an integer \( l \) \((0 \leq l \leq n)\), with \( n + 1 \) even for \( u(t)u^{(n)}(t) \geq 0 \) or \( n + 1 \) odd for \( u(t)u^{(n)}(t) \leq 0 \), and for \( t \geq t_u \),

\[
u(t)u^{(k)}(t) > 0, \quad 0 \leq k \leq l; \quad (-1)^{k+1} u(t)u^{(k)}(t) > 0, \quad l < k \leq n.
\]

**Lemma 1.3** (Philos [16]). Suppose that the conditions of Lemma 1.2 are satisfied, and

\[
u^{(n-1)}(t)u^{(n)}(t) \leq 0, \quad t \geq t_u,
\]
Theorem 2.1. Let 

\[ u'(\frac{1}{2}) \geq M_\theta t^{n-2} |u^{(n-1)}(t)|, \]  

where \( M_\theta = \theta / (n - 2)! \).

2. When \( f(x) \) Is Monotone

In this section, we will deal with the oscillation for (1.1) under the assumptions \((A_1)-(A_6), (S_1)\) and the following assumption.

\((A_6)\) \( f'(x) \) exists, \( f'(x) \geq K_1 \) and \( \varphi(x) \leq L^{-1} \) for \( x \neq 0 \).

Theorem 2.1. Let \((S_1), (A_1)-(A_6)\) hold. Equation (1.1) is oscillatory provided that \( \rho(t) \in C^1([t_0, \infty), R) \) such that

\[ \limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[ H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)\beta}{K_1LM_\theta g^{n-2}(s,a)\varphi'(s,a)} \right] ds = \infty, \]  

where

\[ Q(t) = \rho(t)M \int_{a}^{b} p(t, \xi) f'[1 - q(g(t,\xi))] d\sigma(\xi) - \frac{(\rho'(t))^2 r(t)}{K_1LM_\theta g^{n-2}(t,a)\varphi'(t,a)\rho(t)}. \]  

Proof. Suppose to the contrary that there exists a solution \( x(t) \) of (1.1) such that

\[ x(t) > 0, \ x(t - \tau) > 0, \ x[g(t,\xi)] > 0, \ t \geq t_1, \ \xi \in [a,b] \]  

for \( t \geq t_1 \geq t_0 \).

From (1.1), we also have \( Z(t) > 0 \) and \( [(r(t)\varphi(x(t))Z^{(n-1)}(t))]' \leq 0 \) for \( t \geq t_1 \).

It follows that the function \( r(t)\varphi(x(t))Z^{(n-1)}(t) \) is decreasing and we claim that

\[ Z^{(n-1)}(t) \geq 0 \]  

for \( t \geq t_1 \).

Otherwise, if there exist \( \tilde{t}_1 \geq t_1 \) such that \( Z^{(n-1)}(\tilde{t}_1) < 0 \), then for all \( t \geq \tilde{t}_1 \),

\[ r(t)\varphi(x(t))Z^{(n-1)}(t) \leq r(\tilde{t}_1)\varphi(x(\tilde{t}_1))Z^{(n-1)}(\tilde{t}_1) = -C(C > 0), \]  

which implies that \( Z^{(n-1)}(t) \leq -C/r(t)\varphi(x(t)), \ t \geq t_1 \); integrating the above inequality from \( \tilde{t}_1 \) to \( t \), we have

\[ Z^{(n-2)}(t) \leq Z^{(n-2)}(\tilde{t}_1) - C\int_{\tilde{t}_1}^{t} \frac{1}{r(t)} ds. \]
Let \( t \to \infty \); from \((A_1)\), we get \( \lim_{t \to \infty} Z^{(n-2)}(t) = -\infty \), which implies that \( Z^{(n-1)}(t) \) and \( Z^{(n-2)}(t) \) are negative for all large \( t \); from Lemma 1.2, no two consecutive derivative can be eventually negative, for this would imply that \( \lim_{t \to \infty} Z(t) = -\infty \), which is a contradiction. Hence \( Z^{(n-1)}(t) \geq 0 \) for \( t \geq t_1 \). Using this fact together with \( x(t) \leq Z(t) \), we have that

\[
x(t) \geq [1 - q(t)] Z(t), \quad t \geq t_1.
\]  

(2.7)

Now from \((A_1)\), \((S_1)\), and (2.7), we get

\[
f \left[ x(g(t, \xi)) \right] \geq M f \left[ 1 - q(g(t, \xi)) \right] f \left[ Z(g(t, \xi)) \right], \quad t \geq t_1,
\]  

(2.8)

and thus, from (1.1), we get

\[
0 = \left[ r(t) \psi(x(t)) Z^{(n-1)}(t) \right] + \int_a^b p(t, \xi) f \left[ x(g(t, \xi)) \right] d\sigma(\xi) \geq \left[ (r(t) \psi(x(t)) Z^{(n-1)}(t) \right] \\
+ M \int_a^b p(t, \xi) f \left[ 1 - q(g(t, \xi)) \right] f \left[ Z(g(t, \xi)) \right] d\sigma(\xi).
\]  

(2.9)

Further, observing that \( g(t, \xi) \) is nondecreasing with respect to \( \xi \) and \( Z^{(n-1)}(t) > 0 \) for \( t \geq t_1 \), from Lemma 1.2, we have \( Z'(t) \geq 0, t \geq t_1 \), and so

\[
Z(g(t, \xi)) \geq Z(g(t, a)), \quad t \geq t_1, \quad \xi \in [a, b].
\]  

(2.10)

So, \( f[Z(g(t, \xi))] \geq f[Z(g(t, a))] \) for \( t \geq t_1 \) and \( \xi \in [a, b] \). Thus

\[
\left[ r(t) \psi(x(t)) Z^{(n-1)}(t) \right] + M f \left[ Z(g(t, a)) \right] \int_a^b p(t, \xi) f \left[ 1 - q(g(t, \xi)) \right] d\sigma(\xi) \leq 0, \quad t \geq t_1.
\]  

(2.11)

Define

\[
\omega(t) = \rho(t) \frac{r(t) \psi(x(t)) Z^{(n-1)}(t)}{f[Z(g(t, a)/2)]}, \quad t \geq t_1.
\]  

(2.12)

From (1.1), (2.11), and Lemma 1.3 we get

\[
\omega'(t) = \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \left( \frac{r(t) \psi(x(t)) Z^{(n-1)}(t)}{f[Z(g(t, a)/2)]} \right) \\
- \rho(t) \frac{r(t) \psi(x(t)) Z^{(n-1)}(t)}{f^2[Z(g(t, a)/2)]} f' \left[ Z \left( \frac{g(t, a)}{2} \right) \right] Z' \left( \frac{g(t, a)}{2} \right) \frac{1}{2} g'(t, a) \\
\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) M \int_a^b p(t, \xi) f \left[ 1 - q(g(t, \xi)) \right] d\sigma(\xi) - \frac{1}{2} \frac{K_1 \rho(t) r(t)}{\rho(t) \rho(t) \rho(t)} \omega^2(t).
\]  

(2.13)
Then, by Lemma 1.1 we get

\[
w'(t) \leq -\rho(t) M \int_a^b p(t, \xi) f [1 - q(g(t, \xi))] d\sigma(\xi) + \frac{\rho'(t)^2 r(t)}{K_1 LMg^{(n-2)}(t, a) g'(t, a) \rho(t)}
- \frac{1}{4} \frac{K_1 LMg^{(n-2)}(t, a) g'(t, a)}{\rho(t) r(t)} w^2(t) \tag{2.14}
= -Q(t) - \frac{1}{4} \frac{K_1 LMg^{(n-2)}(t, a) g'(t, a)}{\rho(t) r(t)} w^2(t).
\]

Let

\[
Q(t) = \rho(t) M \int_a^b p(t, \xi) f [1 - q(g(t, \xi))] d\sigma(\xi) - \frac{\rho'(t)^2 r(t)}{K_1 LMg^{(n-2)}(t, a) g'(t, a) \rho(t)}.
\]

That is,

\[
Q(t) \leq -w'(t) - \frac{K_1 LMg^{(n-2)}(t, a) g'(t, a)}{4\rho(t) r(t)} w^2(t). \tag{2.16}
\]

Integrating by parts for any \( t > T \geq t_1 \), and using properties \((A_7)\) and \((A_8)\), we obtain

\[
\int_T^t H(t, s) Q(s) ds
\]
\[
\leq -\int_T^t H(t, s) w'(s) ds - \int_T^t H(t, s) \frac{K_1 LMg^{(n-2)}(s, a) g'(s, a)}{4\rho(s) r(s)} w^2(s)
= H(t, T) w(T) + \int_T^t w(s) \frac{\partial H(t, s)}{\partial s} ds - \int_T^t H(t, s) \frac{K_1 LMg^{(n-2)}(s, a) g'(s, a)}{4\rho(s) r(s)} w^2(s) ds
= H(t, T) w(T) - \int_T^t -w(s) \frac{\partial H(t, s)}{\partial s} ds - \int_T^t H(t, s) \frac{K_1 LMg^{(n-2)}(s, a) g'(s, a)}{4\rho(s) r(s)} w^2(s) ds
= H(t, T) w(T) - \int_T^t \left[ h(t, s) \sqrt{H(t, s) w(s) + H(t, s) \frac{K_1 LMg^{(n-2)}(s, a) g'(s, a)}{4\rho(s) r(s)} w^2(s)} \right] ds
= H(t, T) w(T)
- \int_T^t \left( \sqrt{\frac{H(t, s) K_1 LMg^{(n-2)}(s, a) g'(s, a)}{4\rho(s) r(s)}} \right)^2 ds
+ \frac{\beta}{M_0 K_1 L} \int_T^t g^{(n-2)}(s, a) g'(s, a) ds
- \frac{(\beta - 1) K_1 M_0 L}{4\beta} \int_T^t H(t, s) g^{(n-2)}(s, a) g'(s, a) \frac{w^2(s)}{\rho(s) r(s)} ds. \tag{2.17}
\]
We obtain
\[
\int_t^T \left[ H(t,s)Q(s) - \frac{\beta h^2(t,s)\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)} \right] ds \\
\leq H(t,T)w(T) - \frac{(\beta - 1)K_1 M_0 L}{4\beta} \int_T^t \frac{H(t,s)g^{n-2}(s,a) g'(s,a)}{\rho(s)r(s)} w^2(s) ds \\
- \int_T^t \left( \frac{H(t,s)K_1 L M_0 g^{n-2}(s,a) g'(s,a)}{4\beta \rho(s)r(s)} w(s) + h(t,s) \sqrt{\frac{\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)}} \right)^2 ds. \tag{2.18}
\]

From (A₈), \( H'(t,s) \leq 0 \), for \( t_1 \geq t_0 \), \( H(t,t_1) \leq H(t,t_0) \),
\[
\int_{t_1}^t \left[ H(t,s)Q(s) - \frac{\beta h^2(t,s)\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)} \right] ds \leq H(t,t_1)w(t_1) \leq H(t,t_0)w(t_1), \tag{2.19}
\]
which implies that
\[
\frac{1}{H(t,t_0)} \int_{t_0}^t \left[ H(t,s)Q(s) - \frac{\beta h^2(t,s)\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)} \right] ds \\
\leq w(t_1) + \frac{1}{H(t,t_0)} \int_{t_0}^{t_1} \left[ H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)} \right] ds \tag{2.20}
\]
\[
\leq w(t_1) + \int_{t_0}^{t_1} Q(s) ds < \infty.
\]

Let \( t \to \infty \), and taking upper limits, we have
\[
\lim_{t \to \infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^t \left[ H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)} \right] ds < \infty, \tag{2.21}
\]
which contradicts the assumption (2.1). This complete the proof of Theorem 2.1.

From Theorem 2.1, we have the following oscillation result.

**Corollary 2.2.** If condition (2.1) of Theorem 2.1 is replaced by

\[
\lim_{t \to \infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^t [H(t,s)Q(s)] ds = \infty, \tag{2.22}
\]

\[
\lim_{t \to \infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{h^2(t,s)\rho(s)r(s)}{K_1 L M_0 g^{n-2}(s,a) g'(s,a)} ds < \infty,
\]

where \( Q(t) \) is defined by (2.2), then (1.1) is oscillatory.
Remark 2.3. By introducing various $H(t, s)$ from Theorem 2.1 or Corollary 2.2, we can obtain some oscillatory criteria of (1.1). For example, let $H(t, s) = (t - s)^{m-1}$, $t \geq s \geq t_0$, in which $m > 2$ is a integer. By choosing

$$h(t, s) = (t - s)^{(m-3)/2}(m-1),$$

(2.23)

it is clear that the conditions of $(A_7)$ and $(A_8)$ hold; then, from Theorem 2.1 and Corollary 2.2, we have the following.

Corollary 2.4. Assume that there exists a function $\rho(t) \in C([t_0, \infty), (0, \infty))$ such that

$$\lim_{t \to \infty} \sup_{t_0} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t - s)^{m-1} Q(s) \left[ \frac{1}{\rho(s)} r(s) \right] ds = \infty,$$

(2.24)

where $Q(t)$ is defined by (2.2), then (1.1) is oscillatory.

Corollary 2.5. Assume that there exists a function $\rho(t) \in C([t_0, \infty), (0, \infty))$ such that

$$\lim_{t \to \infty} \sup_{t_0} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t - s)^{m-1} Q(s) ds = \infty,$$

(2.25)

$$\lim_{t \to \infty} \sup_{t_0} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{\rho(s) r(s) \beta}{K_1 L M_0 g^{m-2}(s, a) g'(s, a)} (t - s)^{m-3} (m-1)^2 ds < \infty,$$

where $Q(t)$ is defined by (2.2), then (1.1) is oscillatory.

Theorem 2.6. Assume that the conditions of Theorem 2.1 hold, and

$$0 < \inf_{s \geq t_0} \left( \lim_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} \right) \leq \infty.$$

(2.26)

If there exists a function $\varphi(t) \in C([t_0, \infty), \mathbb{R})$ satisfying

$$\lim_{t \to \infty} \sup_{t_0} \frac{1}{H(t, u)} \int_{u}^{t} \left[ H(t, s) Q(s) - \frac{h^2(t, s) \rho(s) r(s) \beta}{K_1 L M_0 g^{m-2}(s, a) g'(s, a)} \right] ds \geq \varphi(u), \quad u \geq t_0,$$

(2.27)

$$\lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \frac{g^{m-2}(u, a) g'(u, a)}{\rho(u) r(u)} \varphi^2(u) du = \infty, \quad \varphi_*(u) = \max_{u \geq 0} \{ \varphi(u), (0) \},$$

(2.28)

where $Q(t)$ is defined by (2.2), then (1.1) is oscillatory.
Proof. Assume that there exists a nonoscillatory solution \( x(t) \) of (1.1) on \([t_0, \infty)\), such that \( x(t) \neq 0 \) on \([t_0, \infty)\). Without loss of generality, assume that \( x(t) > 0, t \geq t_0 \). Then, proceeding as in the proof of Theorem 2.1, for \( t > u \geq t_1 \geq t_0 \), we have

\[
\frac{1}{H(t, u)} \int_u^t \left[ H(t, s)Q(s) - \frac{\beta h^2(t, s) \rho(s) r(s)}{K_1 LM_0 g^{n-2}(s, a) g'(s, a)} \right] ds \\
geq w(u) - \frac{1}{H(t, u)} \frac{(\beta - 1) K_1 M_0 L}{4\beta} \int_u^t \frac{H(t, s)}{\rho(s) r(s)} g(s, a) g'(s, a) \omega^2(s) ds.
\]  

(2.29)

Let \( t \to \infty \), and taking upper limits, we have

\[
\lim_{t \to \infty} \sup \frac{1}{H(t, u)} \int_u^t \left[ H(t, s)Q(s) - \frac{\beta h^2(t, s) \rho(s) r(s)}{K_1 LM_0 g^{n-2}(s, a) g'(s, a)} \right] ds \\
\leq w(u) - \lim_{t \to \infty} \inf \frac{1}{H(t, u)} \frac{(\beta - 1) K_1 M_0 L}{4\beta} \int_u^t \frac{H(t, s)}{\rho(s) r(s)} g(s, a) g'(s, a) \omega^2(s) ds,
\]

(2.30)

thus, from (2.27), we have

\[
w(u) \geq \varphi(u) + \lim_{t \to \infty} \inf \frac{1}{H(t, u)} \frac{(\beta - 1) K_1 M_0 L}{4\beta} \int_u^t \frac{H(t, s)}{\rho(s) r(s)} g(s, a) g'(s, a) \omega^2(s) ds,
\]

(2.31)

then \( w(u) \geq \varphi(u) \), and

\[
\lim_{t \to \infty} \inf \frac{1}{H(t, u)} \int_u^t \frac{H(t, s)g^{n-2}(s, a) g'(s, a)}{\rho(s) r(s)} \omega^2(s) ds < \frac{4\beta}{(\beta - 1) K_1 M_0 L} (w(u) - \varphi(u)) < \infty.
\]

(2.32)

Now we can claim that

\[
\int_{t_1}^{\infty} \frac{g^{n-2}(s, a) g'(s, a)}{\rho(s) r(s)} \omega^2(s) ds < \infty, \quad t < t_1.
\]

(2.33)

In fact, assume the contrary, that

\[
\int_{t_1}^{\infty} \frac{g^{n-2}(s, a) g'(s, a)}{\rho(s) r(s)} \omega^2(s) ds = \infty, \quad t < t_1.
\]

(2.34)

From (2.26), there exists a constant \( \rho > 0 \) such that

\[
\inf_{s \geq h} \left[ \lim_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} \right] > \rho > 0,
\]

(2.35)
this is
\[
\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} > \rho > 0,
\]
and there exists a \( T_2 \geq t_1 \) such that \( H(t,T)/H(t,t_0) \geq \rho \), for all \( t \geq T_2 \). On the other hand, by virtue of (2.34), for any positive number \( \alpha \), there exists a \( T_1 \geq t_1 \), such that, for all \( t \geq T_1 \)
\[
\int_{t_1}^{t} g^{n-2}(s,a)g'(s,a)\frac{\omega^2(s)}{\rho(s)r(s)}ds > \frac{\alpha}{\rho}.
\]
Using integration by parts, we conclude that, for all \( t \geq T > t_1 \),
\[
\frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)g^{n-2}(s,a)g'(s,a)\frac{\omega^2(s)}{\rho(s)r(s)}ds
\]
\[
= \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)d\left( \int_{t_1}^{s} g^{n-2}(u,a)g'(u,a)\omega^2(s)du \right)
\]
\[
= \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left( \int_{t_1}^{s} g^{n-2}(u,a)g'(u,a)\omega^2(s)du \right) \left( -\frac{\partial H}{\partial s} \right) ds
\]
\[
\geq \frac{\alpha}{\rho} \frac{H(t,T)}{H(t,t_1)} \geq \alpha.
\]
Since \( \alpha \) is an arbitrary positive constant,
\[
\liminf_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)g^{n-2}(s,a)g'(s,a)\frac{\omega^2(s)}{\rho(s)r(s)}ds = \infty,
\]
which contradicts (2.32), consequently, (2.33) holds, and, by virtue of \( \omega(u) \geq \varphi(u) \) for \( u \geq t_1 \geq t_0 \),
\[
\limsup_{t \to \infty} \int_{t_0}^{t} \frac{g'(u,a)g^{n-2}(u,a)}{\rho(u)r(u)}\varphi^2(u)du \leq \limsup_{t \to \infty} \int_{t_0}^{t} \frac{g'(u,a)g^{n-2}(u,a)}{\rho(u)r(u)}\omega^2(u)du < \infty,
\]
which contradicts (2.28), and therefore, (1.1) is oscillatory.

\( \square \)

**Remark 2.7.** Choosing \( H \) as in Remark 2.3, it is not difficult to see that condition (2.26) is satisfied because, for any \( s \geq t_0 \),
\[
\lim_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t \to \infty} \frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1.
\]
Corollary 2.8. Assume that there exist functions \( \rho(t) \in C([t_0, \infty), (0, \infty)) \) and \( \varphi(t) \in C([t_0, \infty), \mathbb{R}) \) satisfying

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \left( (t-s)^{m-1} Q(s) - \frac{\rho(s)r(s)\beta}{K_1 M \theta g^{n-2}(s,a) g'(s,a)} (t-s)^{m-3}(m-1)^2 \right) ds \geq \varphi(u), \quad u \geq 0,
\]

\[
\limsup_{t \to \infty} \int_{t_0}^t \frac{S^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)} \varphi^2(u) du = 0, \quad \varphi_*(u) = \max_{u \geq 0} \{ \varphi(u), (0) \},
\]

where \( Q(t) \) is defined by (2.2), then (1.1) is oscillatory.

Theorem 2.9. Assume that the conditions of Theorem 2.1 and (2.26) hold, and

\[
\liminf_{t \to \infty} \frac{1}{H(t, u)} \int_u^t \left[ H(t, s)Q(s) - \frac{h^2(t, s)\rho(s)r(s)\beta}{K_1 M \theta g^{n-2}(s,a) g'(s,a)} \right] ds \geq \varphi(u), \quad u \geq t_0,
\]

\[
\limsup_{t \to \infty} \int_{t_0}^t \frac{S^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)} \varphi^2(u) du = \infty, \quad \varphi_*(u) = \max_{u \geq t_0} \{ \varphi(u), 0 \},
\]

then (1.1) is oscillatory.

Proof. Assume that there exists a nonoscillatory solution \( x(t) \) of (1.1) on \([t_0, \infty)\), such that \( x(t) \neq 0 \) on \([t_0, \infty)\). Without loss of generality, assume that \( x(t) > 0, t \geq t_0 \). Then, proceeding as in the proof of Theorem 2.1, for \( t > u \geq t_1 \geq t_0 \), we have

\[
\frac{1}{H(t, u)} \int_u^t \left[ H(t, s)Q(s) - \frac{\beta h^2(t, s)\rho(s)r(s)}{K_1 M \theta g^{n-2}(s,a) g'(s,a)} \right] ds \\
\leq w(T) - \frac{1}{H(t, u)} \frac{(\beta - 1)K_1 M \theta L}{4\beta} \int_u^t \frac{H(t, s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)} w^2(s) ds.
\]

Let \( t \to \infty \), and taking lower limits, we have

\[
\liminf_{t \to \infty} \frac{1}{H(t, u)} \int_u^t \left[ H(t, s)Q(s) - \frac{\beta h^2(t, s)\rho(s)r(s)}{K_1 M \theta g^{n-2}(s,a) g'(s,a)} \right] ds \\
\leq w(u) - \limsup_{t \to \infty} \frac{1}{H(t, u)} \frac{(\beta - 1)K_1 M \theta L}{4\beta} \int_u^t \frac{H(t, s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)} w^2(s) ds.
\]

The following proof is similar to Theorem 2.6, so we omit the details. This completes the proof of Theorem 2.9. \( \square \)
3. When $f(x)$ Is Not Monotone

In this section, we will deal with the oscillation for (1.1) under the assumptions $(A_1)$–$(A_8)$ and the following assumption:

\[(A_{10})\] $f(x)/x \geq K_2$ and $\psi(x) \leq L^{-1}$ for $x \neq 0$.

**Theorem 3.1.** Let $(A_1)$–$(A_8)$ and $(A_{10})$ hold. Equation (1.1) is oscillatory provided that $\rho(t) \in C^1([t_0, \infty), R)$ such that

$$
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ H(t, s)Q_2(s) - \frac{h^2(t, s)\rho(s)r(s)\beta}{L \mu g^n(s, a)g'(s, a)} \right] ds = \infty,
$$

(3.1)

where

$$Q_2(t) = \rho(t)K_2 \int_{a}^{b} p(t, \xi) f \left[ 1 - q(g(t, \xi)) \right] d\sigma(\xi) - \frac{(\rho'(t))^2 r(t)}{L \mu g^n(t, a)g'(t, a)\rho(t)}.
$$

(3.2)

then (1.1) is oscillatory.

**Proof.** Let $x(t)$ be an eventually positive solution of (1.1). As in the proof of Theorem 2.1, there exists $t_1 \geq t_0$, such that (2.3), (2.4), and (2.7) hold. Thus, from (1.1) and $(A_{10})$, we get

$$0 = \left( r(t)\psi(x(t))Z^{(n-1)}(t) \right)' + \int_{a}^{b} p(t, \xi) f \left[ x(g(t, \xi)) \right] d\sigma(\xi)
$$

$$\geq \left( r(t)\psi(x(t))Z^{(n-1)}(t) \right)' + K_2 \int_{a}^{b} p(t, \xi) x \left[ g(t, \xi) \right] d\sigma(\xi)
$$

(3.3)

$$= \left( r(t)\psi(x(t))Z^{(n-1)}(t) \right)' + K_2 \int_{a}^{b} p(t, \xi) \left[ Z[g(t, \xi)] - q[g(t, \xi)] x[g(t, \xi) - \tau] \right] d\sigma(\xi).
$$

Noting that

$$Z[g(t, \xi)] \geq Z[g(t, \xi) - \tau] \geq x[g(t, \xi) - \tau].
$$

(3.4)

Thus, (3.3) implies that

$$\left( r(t)\psi(x(t))Z^{(n-1)}(t) \right)' + K_2 \int_{a}^{b} p(t, \xi) \left[ 1 - q(g(t, \xi)) \right] Z[g(t, \xi)] d\sigma(\xi) \leq 0, \quad t \geq t_1.
$$

(3.5)

From (2.10) and (3.5) we get

$$\left( r(t)\psi(x(t))Z^{(n-1)}(t) \right)' + K_2 \int_{a}^{b} p(t, \xi) \left[ 1 - q(g(t, \xi)) \right] d\sigma(\xi) \leq 0, \quad t \geq t_1.
$$

(3.6)
Define
\[ w(t) = \rho(t) \frac{r(t)\psi(x(t))Z^{(n-1)}(t)}{Z[(g(t,a)/2)]}, \quad t \geq t_1. \] (3.7)

Differentiating (3.7) and using (3.6), Lemma 1.1, and 1.3 we get
\[
\begin{align*}
w'(t) & \leq \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t) \left[ K_2 \int_a^b p(t, \xi) \{1 - q(g(t, \xi))\} d\sigma(\xi) \right] \\
& \quad - \frac{M_0 L g^{n-2}(t, a) g'(t, a)}{4\rho(t)r(t)} w^2(t) \\
& \quad - \frac{M_0 L g^n(t, a) g'(t, a)}{4\rho(t)r(t)} w^2(t) \\
& = -Q_2(t) - \frac{M_0 L g^n(t, a) g'(t, a)}{4\rho(t)r(t)} w^2(t).
\end{align*}
\] (3.8)

The rest proof is similar to that of Theorem 2.1 and hence is omitted. This completes the proof of Theorem 3.1.

**Theorem 3.2.** Assume that the conditions of Theorem 2.1 and (2.26) hold; if there exists a function \( \varphi(t) \in C([t_0, \infty), \mathbb{R}) \) satisfying
\[
\limsup_{t \to \infty} \frac{1}{H(t, u)} \int_u^t \left[ H(t, s)Q_2(s) - \frac{h^2(t, s)\rho(s)r(s)\beta}{LM_0 g^{n-2}(s, a) g'(s, a)} \right] ds \geq \varphi(u), \quad u \geq t_0,
\]
\[
\limsup_{t \to \infty} \int_{t_0}^t \frac{g^n(u, a) g'(u, a)}{\rho(u)r(u)} \varphi^2(u) du = \infty, \quad \varphi_+(u) = \max_{u \geq t_0} \{\varphi(u), 0\},
\] (3.9)
then (1.1) is oscillatory.

**Theorem 3.3.** Let all assumptions of Theorem 2.6 be satisfied except that \( \limsup \) in condition Theorem 3.2 is replaced with \( \liminf \), then (1.1) is oscillatory.

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**References**


