A Note on the Global Attractivity of a Discrete Model of Nicholson’s Blowflies*

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(Received 4 February 1999)

In this paper, we further study the global attractivity of the positive equilibrium of the discrete Nicholson’s blowflies model

$$N_{n+1} - N_n = -\delta N_n + pN_{n-k}e^{-aN_n}, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (1)

We obtain a new criterion for the positive equilibrium $N^*$ to be a global attractor, which improve the corresponding results obtained by So and Yu (J. Math. Anal. Appl. 193 (1995), 233–244).

Keywords: Attractivity, Positive equilibrium, Discrete Nicholson’s blowflies model

AMS Subject Classification: 39A10

I. INTRODUCTION

The delay difference equation

$$N_{n+1} - N_n = -\delta N_n + pN_{n-k}e^{-aN_n}, \quad n = 0, 1, 2, \ldots,$$  \hspace{1cm} (1)

is a discrete analogue of the delay differential equation

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}, \quad t \geq 0,$$

which has been used in describing the dynamics of Nicholson’s blowflies [2,4–6].

By the biology consideration, we assume that $\delta \in (0, 1)$, $p, a \in (0, +\infty)$, and $k \in N = \{0, 1, 2, \ldots\}$. The initial condition is

$$N_j = \varphi_j \geq 0, \quad j \in \{-k, -k+1, \ldots, 0\},$$  \hspace{1cm} (2)

and $\varphi_j > 0$, for some $j \in \{-k, -k+1, \ldots, 0\}$.

By a solution of (1) and (2) we mean a sequence $\{N_n\}$ which satisfies (1) for $n = 0, 1, 2, \ldots$ as well as the initial condition (2). Clearly, the unique solution $\{N_n\}$ of the above initial value problem is positive for all large $n$ [1].

* This work is supported by NNSF of China.

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If $p > \delta$, then Eq. (1) has a unique positive equilibrium $N^*$ and
\[ N^* = \frac{1}{a} \ln \left( \frac{p}{\delta} \right). \]  
(3)

The global attractivity of $N^*$ was studied by Kocic and Lada [3] and So and Yu [1] respectively. The recent result is the following [1].

**THEOREM A**  Assume that $p > \delta$ and that
\[ \left( (1 - \delta)^{-k - 1} - 1 \right) \ln \left( \frac{p}{\delta} \right) \leq 1. \]  
(4)

Then any nontrivial solution $N_n$ of (1) and (2) satisfies
\[ \lim_{n \to \infty} N_n = N^*. \]

In this note, our purpose is to improve condition (4). Exactly speaking, we will show some conditions for the global attractivity of $N^*$ when (4) does not hold. Our results are discrete analogues of the results in [2].

To prove our main results, we need some known results.

**LEMMA 1** [1]  Let $\{N_n\}$ be a solution of (1) and (2). Then
\[ \limsup_{n \to \infty} N_n \leq \frac{p}{ae^\delta}. \]  
(5)

As in [2], the following system of inequalities
\[ \begin{cases} y + \ln(1 + (y/aN^*)) \leq M(e^{-x} - 1), \\ x + \ln(1 + (x/aN^*)) \geq M(e^{-y} - 1), \end{cases} \]  
(6)

play an important role in our analysis, where $M = aN^*[((1 - \delta)^{-k - 1} - 1) = ((1 - \delta)^{-k - 1} - 1) \ln(p/\delta)$. Let
\[ D = \{(x, y): -aN^* < x \leq 0 \leq y < \infty\}. \]  
(7)

**LEMMA 2** [2]  If one of the following conditions holds:
(i) $M \leq 1$;
(ii) $M < 1 + (1/aN^*)$ and $aN^* \geq (\sqrt{3} - 1)/2$;
(iii) $M \leq 1 + (1/aN^*)$ and $aN^* > (\sqrt{1 + 4\sqrt{3} - 1})/2$,
then (6) has a unique solution $x = y = 0$ in $D$.

**II. MAIN RESULTS**

The following theorem provides a new sufficient condition for the equilibrium $N^* = (1/a)\ln(p/\delta)$ to be a global attractor.

**THEOREM 1**  Assume that $p > \delta$ and the assumption in Lemma 2 holds. Then any nontrivial solution $\{N_n\}$ of (1) and (2) satisfies
\[ \lim_{n \to \infty} N_n = N^*. \]

**Proof**  Let
\[ N_n = N^* + \frac{1}{a} x_n. \]

Then $\{x_n\}$ is a solution of the equation
\[ x_{n+1} - x_n + \delta x_n + a\delta N^*(1 - e^{-x_{n-k}}) - \delta e^{-x_{n-k}} = 0, \quad n = 0, 1, 2, \ldots \]  
(8)

Since $N_n > 0$ for all large $n$, it follows that $x_n > -aN^*$ for all large $n$.

To prove this theorem, it is sufficient to prove
\[ \lim_{n \to \infty} x_n = 0. \]  
(9)

Let
\[ \mu = \limsup_{n \to \infty} x_n \quad \text{and} \quad \lambda = \liminf_{n \to \infty} x_n. \]

Then $-aN^* \leq \lambda \leq \mu < \infty$. We claim that $\lambda = \mu = 0$. For the case $\{x_n\}$ is eventually nonnegative or eventually nonpositive, this has been proved in the proof of Theorem 2 in [3]. Therefore it is sufficient to consider the case that $\{x_n\}$ is an oscillatory solution of (8).

Our purpose is to prove that $\lambda = \mu = 0$ under the assumptions. There are four possible cases:
(i) $\lambda = \mu = 0$;
(ii) $\mu > 0$ and $\lambda = 0$;
The cases 2 and 3 can be considered to be special cases of case 4. Now we consider case 4.

In this case, there exists a sequence \(\{n_i\}\) of positive integers such that

\[
k < n_1 < n_2 < \cdots < n_i < n_{i+1} \to \infty \quad \text{as} \quad i \to \infty.
\]

\[
x_{n_i} < 0 \quad \text{and} \quad x_{n_{i+1}} \geq 0, \quad \text{for} \quad i = 1, 2, \ldots,
\]

and for each \(i = 1, 2, \ldots\), the terms of the finite sequence \(x_j\) for \(n_i < j < n_{i+1}\) assume both positive and negative values. Let \(m_i\) and \(M_i\) be integers in \((n_i, n_{i+1})\) such that for \(i = 1, 2, \ldots\)

\[
x_{M_i} = \max\{x_j; \quad n_i < j < n_{i+1}\},
\]

and

\[
x_{m_i} = \min\{x_j; \quad n_i < j < n_{i+1}\}.
\]

We can assume without loss of generality that for \(i = 1, 2, \ldots\)

\[
x_{M_i} > 0, \quad x_{M_i} - x_{M_{i-1}} \geq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{M_i} = \mu > 0,
\]

while

\[
x_{m_i} < 0, \quad x_{m_i} - x_{m_{i-1}} \leq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{m_i} = \lambda < 0.
\]

Then there exist subsequence \(\{q_i\}\) of \(\{m_i\}\) and subsequence \(\{Q_i\}\) of \(\{M_i\}\) such that

\[
x_{Q_i} > 0, \quad x_{Q_i} - x_{Q_{i-1}} \geq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{Q_i} = \mu > 0,
\]

while

\[
x_{q_i} < 0, \quad x_{q_i} - x_{q_{i-1}} \leq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{q_i} = \lambda < 0.
\]

It follows from (8) and (10) that

\[
x_{Q_{i-1}} + aN^* \leq [x_{Q_{i-k-1}} + aN^*]e^{-x_{Q_{i-k-1}}},
\]

thus

\[
x_{Q_i} + aN^* = (1 - \delta)(x_{Q_{i-1}} + aN^*) + \delta(x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}}
\]

\[
\leq (1 - \delta)(x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}}
\]

\[
+ \delta(x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}}
\]

\[
= (x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}}
\]

that is

\[
x_{Q_i} + aN^* \leq (x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}}. \quad (12)
\]

Now let us prove

\[
x_{Q_{i-k-1}} < 0,
\]

assume the contrary, then \(x_{Q_{i-k-1}} = 0\) or \(x_{Q_{i-k-1}} > 0\). If \(x_{Q_{i-k-1}} = 0\), then \(x_{Q_i} \leq 0\), which contradicts (10). If \(x_{Q_{i-k-1}} > 0\), then \(x_{Q_{i-k-1}} > x_{Q_i}\), thus

\[
\lim_{i \to \infty} x_{Q_{i-k-1}} \leq \lim_{i \to \infty} x_{Q_i} = \mu,
\]

on the other hand, we have

\[
\lim_{i \to \infty} x_{Q_{i-k-1}} \leq \lim_{i \to \infty} x_{M_i} = \mu,
\]

so we get

\[
\lim_{i \to \infty} x_{Q_{i-k-1}} = \mu, \quad (14)
\]

then taking the limit in (12), we obtain

\[
\mu + aN^* \leq (\mu + aN^*)e^{-\mu},
\]

which implies \(\mu \leq 0\) that contradicts (10), so (13) holds.

From (12) and (13), we have

\[
x_{Q_i} + aN^* < aN^*e^{-x_{Q_{i-k-1}}},
\]
therefore
\[ x_{Q_i-k-1} < -\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right). \] (15)

For given \(\varepsilon > 0\), by (9), there exists a positive integer \(n^*\) such that
\[ \lambda - \varepsilon < x_n < \mu + \varepsilon, \quad \text{for} \ n \geq n^* - k, \]
this induce \(x_{n-k}e^{-x_{n-k}} < \mu + \varepsilon\), for \(n \geq n^*\).

Rewriting Eq. (8) into the following form:
\[
(1 - \delta)^{-n-1}x_{n+1} - (1 - \delta)^{-n}x_n + a\delta N^*(1 - \delta)^{-n-1}(1 - e^{-x_{n-k}}) - \delta(1 - \delta)^{-n-1}x_{n-k}e^{-x_{n-k}} = 0. \] (16)

Now summing (16) up from \(n = Q_i - k - 1\) (assuming \(Q_i - k - 1 \geq n^*\)) to \(n = Q_i - 1\), we have
\[
(1 - \delta)^{-Q_i}x_{Q_i} = (1 - \delta)^{-Q_i-k+1}x_{Q_i-k} - a\delta N^* \\
\times \sum_{n=Q_i-k}^{Q_i-1} (1 - \delta)^{-n-1}(1 - e^{-x_{n-k}}) + \delta \sum_{n=Q_i-k}^{Q_i-1} (1 - \delta)^{-n-1}x_{n-k}e^{-x_{n-k}} < (1 - \delta)^{-Q_i-k+1}x_{Q_i-k} + a\delta N^* \\
\times \sum_{n=Q_i-k}^{Q_i-1} (1 - \delta)^{-n-1}(e^{-\lambda + \varepsilon} - 1) + \delta \sum_{n=Q_i-k}^{Q_i-1} (1 - \delta)^{-n-1}(\mu + \varepsilon) \\
= (1 - \delta)^{-Q_i-k+1}x_{Q_i-k} \\
+ [((\mu + \varepsilon) + aN^*(e^{-\lambda + \varepsilon} - 1)] \\
\times (1 - \delta)^{-Q_i}[1 - (1 - \delta)^{k+1}],
\]
Substituting (15) into the above inequality, we get
\[
(1 - \delta)^{-Q_i}x_{Q_i} < -((1 - \delta)^{-Q_i-k+1}\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right) \\
+ [((\mu + \varepsilon) + aN^*(e^{-\lambda + \varepsilon} - 1)] \\
\times (1 - \delta)^{-Q_i}[1 - (1 - \delta)^{k+1}],
\]
and
\[
x_{Q_i} + (1 - \delta)^{k+1}\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right) < [[\mu + \varepsilon] + aN^*(e^{-\lambda + \varepsilon} - 1)]\ln\left(1 - (1 - \delta)^{k+1}\right),
\]
let \(i \to \infty, \varepsilon \to 0\), we get
\[
\mu + (1 - \delta)^{k+1}\ln\left(1 + \frac{\mu}{aN^*}\right) \\
\leq [\mu + aN^*(e^{-\lambda} - 1)]\ln\left(1 - (1 - \delta)^{k+1}\right).
\]
We rewrite the above inequality:
\[
\mu + \ln\left(1 + \frac{\mu}{aN^*}\right) = M(e^{-\lambda} - 1). \] (17)
In a similar way, we have
\[
\lambda + \ln\left(1 + \frac{\lambda}{aN^*}\right) = M(e^{-\mu} - 1). \] (18)
Then we establish the following system of inequalities:
\[
\left\{ \begin{array}{l}
\mu + \ln(1 + \frac{\mu}{aN^*}) \leq M(e^{-\lambda} - 1), \\
\lambda + \ln(1 + \frac{\lambda}{aN^*}) \geq M(e^{-\mu} - 1).
\end{array} \right. \] (19)
For case 2, the system of inequalities corresponding to (19) is
\[
\left\{ \begin{array}{l}
\mu + \ln(1 + \frac{\mu}{aN^*}) \leq M(e^{-\lambda} - 1), \\
\lambda = 0.
\end{array} \right. \] (20)
It is obvious that (20) holds iff \(\lambda = \mu = 0\).
For case 3, the system of inequalities corresponding to (19) is
\[
\left\{ \begin{array}{l}
\mu = 0, \\
\lambda + \ln(1 + \frac{\lambda}{aN^*}) \geq M(e^{-\mu} - 1).
\end{array} \right. \] (21)
Similarly, (21) holds iff \(\lambda = \mu = 0\).
Thus it will suffice to consider case 4, for (19) in case 4, by Lemma 2, we get \(\lambda = \mu = 0\). So the proof is complete.
Remark 1 In cases 2 and 3 in Theorem 1, we add some reasonable conditions to $aN^*$. We know
\[ M = aN^*[(1 - \delta)^{-k-1} - 1] \leq 1 + \frac{1}{aN^*}, \]
on the right side of which there is nothing to do with $\delta$ and $k$. While \(1 + (1/aN^*) \to \infty\) as $aN^* \to 0^+$, properly choosing the values of $[(1 - \delta)^{-k-1} - 1]$, we can let $M$ equal or infinitely tend to the value of $1 + (1/aN^*)$, then $M$ can be changed to arbitrarily large. Obviously this is not reasonable.

Remark 2 Theorem 4.1 in [1] only applies to the case $M \leq 1$, while Theorem 1 in this paper not only applies to $M \leq 1$ but also to $M > 1$. So the results in this paper improve those in [1].

Example Consider the delay difference equation
\[ N_{n+1} - N_n = -\frac{1}{4}N_n + \frac{1}{4}e^{(\sqrt{5} - 1)/2}N_{n-3}e^{-2N_{n-3}}, \quad (22) \]
then we can calculate
\[ aN^* = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad [(1 - \delta)^{-k-1} - 1] = \frac{175}{81}, \]
thus,
\[ M \approx 1.335 \quad \text{and} \quad 1 + \frac{1}{aN^*} = \frac{\sqrt{5} + 3}{2} \approx 2.618. \]
The conditions in Theorem 1 are satisfied. Thus
\[ N^* = \frac{\sqrt{5} - 1}{4} \]
is a global attractor or (22). But Theorem 4.1 in [1] cannot apply to this case.

References