Research Article

Restricted \( p \)-Isometry Properties of Partially Sparse Signal Recovery

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By generalizing the restricted \( p \)-isometry property to the partially sparse signal recovery problem, we give a sufficient condition for exactly recovering partially sparse signal via the partial \( l_p \) minimization (truncated \( l_p \) minimization) problem with \( p \in (0,1] \).

Based on this, we establish a simpler sufficient condition which can show how the \( p \)-RIP bounds vary corresponding to different \( p \)s.

1. Introduction

The partially sparse signal recovery (PSSR) is the problem of recovering a partially sparse signal from a certain number of linear measurements when the part of the signal is known to be sparse, which was coined by Bandeira et al. [1, 2]. This type of problems has many applications in signal and image processing, derivative-free optimizations, and so on; see, for example, [1–4]. Clearly, PSSR includes sparse signal recovery (SSR) as a special case. The latter is the well-known NP-hard problem in the compressed sensing (CS), which is also called cardinality minimization problem (CMP; or \( l_0 \)-norm minimization problems); see, for example, [5–8].

In particular, Candès and Tao [8] introduced a restricted isometry property (RIP) of a sensing matrix which guarantees to recover a sparse solution of SSR by minimizing its convex relaxation (\( \ell_1 \)-norm minimization). However, there are some problems which cannot be reformulated as an SSR, but a PSSR. As we know, PSSR happens naturally in sparse Hessian recovery; see, for example, [2], where Bandeira et al. employed partially sparse recovery approach for building sparse quadratic interpolation models of functions with sparse Hessian. They have successfully applied the \( \ell_1 \)-norm minimization of PSSR in interpolation-based trust-region methods for derivative-free optimization. Vaswani and Lu [3] successfully applied modified CS (partially sparse recovery) in image reconstruction, where the sufficient RIP condition is weaker than the RIP for SSR. Moreover, Bandeira et al. [1] considered the RIP and null space properties (NSP) for PSSR and extended recovery results under noisy measurements to the partially sparse case, where partial NSP is a necessary and sufficient condition for PSSR. In [4], Jacques also established the partial RIP condition for PSSR with noise via its convex relaxation problem.

Note that in the CS context, the SSR problem can also be relaxed to a \( l_p \)-norm minimization (truncated \( l_p \)-minimization) problem with \( 0 < p < 1 \); see, for example, [9–19]. It is well known that Chartrand [20] firstly show that fewer measurements are required for exact reconstruction if we replace \( l_1 \)-norm with \( l_p \)-norm \( (0 < p < 1) \), and Chartrand and Staneva [10] established \( p \)-RIP conditions for exact SSR via \( l_p \)-minimization. In particular, the numerical experiments in magnetic resonance imaging (MRI) showed that this approach works very efficiently; see [9] for details. Wang et al. [19] studied the performance of \( l_p \)-minimization for strong recovery and weak recovery where we need to recover all the sparse vectors on one support with one sign pattern. Moreover, Saab et al. [16] provided a sufficient condition for SSR via \( l_p \)-minimization and provided a lower bound of the support size up to which \( l_p \)-minimization can recover all such sparse vectors, and Foucart and Lai [14] improved this bound by considering a generalized version of RIP condition.

While SSR and \( l_p \)-minimization have been the focus point of some recent research, there are fewer research related to PSSR and the partially \( l_p \)-minimization. One may naturally wonder whether we can generalize the \( p \)-RIP conditions...
introduced by [10] from the SSR to the PSSR case. This paper will deal with this issue. We will give a different p-RIP recovery condition for PSSR via its nonconvex relaxation. Furthermore, based on the recent work by Oymak et al. [21], we also extend our result to the matrix setting.

In the next section, we give the PSSR model and review some preliminaries on p-RIP conditions. In Section 3, we establish the exact partially p-RIP recovery conditions for PSSR via its nonconvex $l_p$-minimization. In Section 4, we give a sufficient condition for partially low-rank matrix recovery via the partially Schatten-$p$ minimization problem.

2. Preliminaries

In this section, we will review some basic concepts and results on the p-RIP recovery conditions for SSR and introduce the p-RIP definition for PSSR. We begin with defining the mathematical model of the PSSR problem as follows:

$$\min \|x\|_0, \quad \text{s.t. } A_1 x + A_2 y = b, \quad (1)$$

where the $l_0$-norm $\|x\|_0$ is defined as $\|x\|_0 := |\{ i : x_i \neq 0 \}|$ (which is not really a norm since it is not positive homogeneous). For any positive number $s$, we say $x$ is $s$-sparse if $\|x\|_0 \leq s$. $A = [A_1, A_2] \in \mathbb{R}^{M \times N}$ is a sensing matrix with $A_1 \in \mathbb{R}^{M \times (N-r)}$, $A_2 \in \mathbb{R}^{M \times r}$, and $b \in \mathbb{R}^M$. It means that the unknown vector consists of two parts $(x, y)$, where $x \in \mathbb{R}^{N-r}$ is sparse and $y \in \mathbb{R}^r$ is possibly dense. When $A = A_1$, the previous problem reduces to the following $l_0$-norm minimization problem (sparse signal recovery, SSR):

$$\min \|x\|_0, \quad \text{s.t. } A x = b. \quad (2)$$

The previous PSSR problem (1) is an NP-hard problem, since its special case SSR (2) is well-known NP-hard problem in the compressed sensing (CS). As we mentioned in Section 1, one popular and powerful approach is to solve it via $l_1$-norm minimization (its convex relaxation), where the $l_0$-norm is replaced by the $l_1$-norm in SSR (2). Moreover, we can also use a nonconvex approach for exact reconstruction with fewer measurements than the convex relaxation; see, for example, [9, 10]. That is the $l_p$-norm minimization problem with $0 < p < 1$, where we replace the $l_0$-norm with the $l_p$-norm in (2) as follows:

$$\min \|x\|_p^p, \quad \text{s.t. } A x = b. \quad (3)$$

Note that $\| \cdot \|_p$ is not a norm when $p \in (0, 1)$, but it is much close to $l_0$-norm. Moreover, the numerical experiments in MRI showed that the approach via $l_p$-minimization works very efficiently; see [9] for details. In particular, Chartrand and Staneva [10] introduced the concept of restricted isometry constant via $l_p$-norm.

**Definition 1 (p-RIC [10]).** Given a matrix $A \in \mathbb{R}^{M \times N}$, where $M < N$, $s$ is a positive number and $0 < p < 1$, then we say that $\delta_p$ is the restricted $p$-isometry constant (or p-RIC) of order $s$ of the matrix $A$ if $\delta_p$ is the smallest number such that

$$(1 - \delta_p) \|x\|_2^2 \leq \|Ax\|_p^p \leq (1 + \delta_p) \|x\|_2^2, \quad (4)$$

for all $s$-sparse vectors $x$.

In the same paper, Chartrand and Staneva gave the following sufficient condition for exact SSR via $l_p$-minimization.

**Theorem 2** (see [10]). Let $A \in \mathbb{R}^{M \times N}$, $x \in \mathbb{R}^N$, and $k = \|x\|_0$ be the size of the support of $x$, $0 < p < 1$, $a_1 > 1$, and $a_2 = a_1^{\frac{1}{2}(1-p)}$, rounded up, so that $a_2 k$ is an integer ($a_2 = \lceil a_1^{\frac{1}{2}(1-p)} k \rceil / k$). If $A$ satisfies

$$\delta_{a_2 k} A \delta_{a_2 k}^t A = a_1 - a_2, \quad \delta_{a_2 k}^t A \delta_{a_2 k} A = a_1 - a_2, \quad \delta_{a_2 k} A \delta_{a_2 k}^t A = a_1 - a_2, \quad (5)$$

then $x$ is the unique minimizer of problem (2).

Inspired by the previous analysis, it is natural to give the partially $l_p$-norm minimization problem for PSSR (1) as follows:

$$\min \|x\|_p^p, \quad \text{s.t. } A_1 x + A_2 y = b. \quad (6)$$

In order to establish the link between the PSSR (1) and its partially $l_p$-norm minimization problem, we need to give a partially $p$-RIC definition. Here we borrow the idea from Bandeira et al. [1]. Assume that $A_2$ is full column rank. For $A = [A_1, A_2]$ as mentioned above, let

$$B := I - A_2 (A_2^T A_2)^{-1} A_2^T, \quad (7)$$

which is the matrix of the orthogonal projection from $\mathbb{R}^N$ to $\mathbb{R}(A_2)^\perp$.

**Definition 3** (Partially $p$-RIC). Let $A = [A_1, A_2] \in \mathbb{R}^{M \times N}$, where $A_1 \in \mathbb{R}^{M \times (N-r)}$, and $A_2 \in \mathbb{R}^{M \times r}$ is full column rank. We say that $\delta_{r}^f$ is the Partially Restricted Isometry Constant (Partially $p$-RIC) of order $s-r$ of the matrix $A$ if $\delta_{r}^f$ is the $p$-RIC of order $s-r$ of the matrix $BA_1$; that is, $\delta_{r}^f$ is the smallest number such that

$$(1 - \delta_{r}^f) \|x\|_2^2 \leq \|BA_1 x\|_p^p \leq (1 + \delta_{r}^f) \|x\|_2^2, \quad (8)$$

for all $(s-r)$-sparse vectors $x$, where $B$ is given by (7).

3. Main Results

We will give our main results which state sufficient $p$-RIP recovery conditions on the exact PSSR via the nonconvex $l_p$-norm minimization. We begin with the following useful lemma.

**Lemma 4.** For $0 < p \leq 1$, let $c_1 = a_1^{1-p/2}$ and $c_2 = a_2^{1-p/2}$ with $a_1 > 0$ and $a_2 > 0$. If $(c_1 - 1)/c_2 > |a_1 - a_2|/a_2$, then $a_1 > 1$ and $a_2 > 1$.

*Proof.* In order to prove the lemma, we consider the following two cases.

**Case 1** $(a_1 \geq a_2)$. In this case, from the fact $(c_1 - 1)/c_2 > |a_1 - a_2|/a_2$, we have

$$\frac{a_1}{a_2} - \left( \frac{a_1}{a_2} \right)^{1-p/2} < 1 - \left( \frac{1}{a_2} \right)^{1-p/2}. \quad (9)$$
If \( 0 < a_2 \leq 1 \), from the previous inequality we easily obtain
\[
0 < a_1 \frac{1}{a_2} - (a_1 \frac{1}{a_2})^{1/p^2} < 1 - \left( \frac{1}{a_2} \right)^{1/p^2} \leq 0,
\]
which is a contradiction. Hence \( a_1 \geq a_2 > 1 \).

**Case 2** \( a_1 < a_2 \). Similarly, in this case, from \( (c_1 - 1)/c_2 > |a_1 - a_2|/a_2 \), we obtain
\[
0 < 1 - \frac{a_1}{a_2} < \frac{c_1 - 1}{c_2}.
\]

If \( 0 < a_1 \leq 1 \), then
\[
c_1 = a_1^{1/p^2} \leq 1.
\]
Combining the previous inequalities we obtain
\[
0 < 1 - \frac{a_1}{a_2} < \frac{c_1 - 1}{c_2} \leq 0,
\]
which is a contradiction. Hence \( 1 < a_1 < a_2 \).

Therefore, taking into account the previous two cases, we completed the proof.

We below propose a general recovery condition for PSSR via its \( p \)-norm minimization.

**Theorem 5.** Let \( A = [A_1, A_2] \in \mathbb{R}^{M \times N} \) with \( A_1 \in \mathbb{R}^{M \times (N-r)} \) and \( A_2 \in \mathbb{R}^{M \times r} \). Suppose that \( A_2 \) is full column rank, and let \( A_1 x + A_2 y = b \) with \( \|x\|_0 = k \). For \( 0 < p \leq 1 \), \( a_1 > 0 \), and \( a_2 > 0 \), let \( c_1 = a_1^{1-p/2} \), \( c_2 = a_2^{1-p/2} \) with \( (c_1-1)/c_2 > |a_1-a_2|/a_2 \).

If
\[
a_2 c_1 \Delta_{(a_1+1)k} + (|a_1 - a_2|/c_2) \Delta_{a_1 k} < a_2 c_1 - |a_1 - a_2|/c_2 - a_2,
\]
then \( (x, y) \) is the unique minimizer of problem (6).

**Proof.** Note that \( (x, y) \) is a feasible solution to optimization problem (6). We remain to show that the solution set is a singleton \( \{ (x, y) \} \). This proof generally modifies that of [10], but under different assumptions. (Specifically, we use a different way to arrange the elements of \( T_0^C \) in the following.) Let \( (u, v) \) be an arbitrary solution to problem (6). We will show that \( u = x \) and \( y = v \). We will prove \( u = x \) firstly. First, \( h = u - x \), we will show that \( h = 0 \). Let \( \Phi = BA_1 \). For \( T \in \{ 1, \ldots, N - r \} \), \( \Phi_T \) denotes the matrix equaling \( \Phi \) in those columns whose indices belong to \( T \) and otherwise zero. Similarly, we define the vector \( h_T \). Let \( T_0 \) be the support of \( x \). Then, the supports of \( x \) and \( h_T^C \) are disjoint since \( T_0 \cap T_0^C = \emptyset \). From direct calculation, we obtain
\[
\|x\|_p^p \geq \|u\|_p^p = \|x + h\|_p^p
\]
\[
= \|x + h_{T_0} + h_{T_0^C}\|_p^p
\]
\[
= \|x + h_{T_0}\|_p^p + \|h_{T_0^C}\|_p^p
\]
\[
\geq \|x\|_p^p - \|h_{T_0}\|_p^p + \|h_{T_0^C}\|_p^p,
\]
where the first inequality holds because \((u, v)\) solves (6), and the last one holds by the triangle inequality for \( \| \cdot \|_p \). Then we have
\[
\|h_{T_0}\|_p^p \leq \|h_{T_0^C}\|_p^p. \tag{16}
\]

Now we arrange the elements of \( T_0^C \) in order of decreasing magnitude of \( |h| \) and partition into \( T_0^C = T_1 \cup T_2 \cup \cdots \cup T_p \), where \( T_1 \) has \( a_2 k \) elements and \( T_j \) \( (j \geq 2) \) each has \( a_1 k \) elements (except possibly \( T_j \)). Set \( T_{01} = T_0 \cup T_1 \). Note that
\[
BA_2 = \left[ I - A_2 (A_2^T A_2)^{-1} A_2^T \right] A_2 = 0. \tag{17}
\]

Direct calculations yield
\[
0 = \|B (A_1 x + A_2 y - A_1 u - A_2 v)\|_p^p
\]
\[
= \|BA_1 x - BA_1 u\|_p^p = \|\Phi x - \Phi u\|_p^p
\]
\[
= \|\Phi h_T\|_p^p = \|\Phi h_{T_0} + \sum_{j \geq 2} \Phi h_{T_j}\|_p^p
\]
\[
\geq \|\Phi h_{T_0}\|_p^p \geq \|\Phi h_{T_0}\|_p^p - \sum_{j \geq 2} \|\Phi h_{T_j}\|_p^p
\]
\[
\geq \left( 1 - \Delta_{(a_1+1)k} \right) \|h_{T_0}\|_p^p - \left( 1 - \Delta_{a_1 k} \right) \|h_{T_0}\|_p^p
\]
\[
\times \sum_{j \geq 2} \|h_{T_j}\|_p^p.
\]

Now we discuss the relation between \( \ell_2 \)-norm and \( \ell_p \)-norm. For each \( t \in T_j \) and \( s \in T_{j-1} \), it holds \( |h| \leq |h| \). So, we have for \( j = 2, \)
\[
|h_t|^p \leq \frac{\|h_{T_j}\|_p}{a_2 k}
\]
\[
\Rightarrow |h_t|^2 \leq \frac{\|h_{T_j}\|_p^2}{(a_2 k)^{2/p}}
\]
\[
\Rightarrow |h_{T_0}|^2 \leq \frac{\|h_{T_0}\|_p^2}{(a_2 k)^{2/p}}
\]
\[
\Rightarrow \|h_{T_0}\|_p^p \leq \frac{(a_2 k)^{2/p}}{a_1 k} \|h_{T_0}\|_p^p.
\]
Similarly, we obtain that for \( j \geq 3 \),
\[
\|h_{T_j}\|^p \leq \frac{1}{(a_k k)^{1-p/2}} \|h_{T_{j-1}}\|^p.
\]
(20)

Applying the Holder's inequality, we obtain
\[
\|h_{T_1}\|^p = \sum_{t \in T_1} |h_t|^p \cdot 1
\]
\[
\leq \left( \sum_{t \in T_1} |h_t|^2 \right)^{p/2} \left( \sum_{t \in T_1} 1 \right)^{1-p/2}
\]
\[
= \|h_{T_1}\|^p \cdot k^{1-p/2}.
\]
(21)

Similarly, we have
\[
\|h_{T_1}\|^p \leq \|h_{T_1}\|^p \cdot (a_k k)^{1-p/2}.
\]
(22)

Therefore,
\[
\sum_{j \geq 2} \|h_{T_j}\|^p \leq \frac{a^2}{a_k} \sum_{j \geq 2} \|h_{T_j}\|^p.
\]
(23)

Thus by (18) and (23), we have
\[
0 \geq (1 - \delta_{(a+1)k}) \|h_{T_{j-1}}\|^p
\]
\[
- \left(1 + \delta_{a} \right) \sum_{j \geq 2} \|h_{T_j}\|^p \quad \text{(By (18))}
\]
\[
\geq (1 - \delta_{(a+1)k}) \|h_{T_{j-1}}\|^p - \frac{1 + \delta_{a}}{c_1} \|h_{T_{j-1}}\|^p
\]
\[
= \left[ 1 - \delta_{(a+1)k} - \frac{1 + \delta_{a}}{c_1} \frac{(a_1 - a_2) c_2}{a_2 c_1} (1 + \delta_{a}) \right]
\]
\[
\times \|h_{T_{j-1}}\|^p.
\]
(24)

Clearly, the assumption ensures that the scalar factor is positive, and hence we obtain \( h_{T_{j-1}} = 0 \). That means \( h_{T_1} = 0 \).

Using \( \|h_{T_1}\|^p \leq \|h_{T_{j-1}}\|^p \), we obtain \( h_{T_1} = 0 \). Therefore, \( h = 0 \), which means \( x = u \).

Now we remain to show that \( y = v \). It is obvious that \( A_1 x + A_2 y = A_1 u + A_2 v \). Since \( x = u \), we have \( A_2 (y - v) = 0 \).

Then \( y = v \) because \( A_2 \) is full column rank.\]

Theorem 5 states a different sufficient condition for the exactly PSSR via its nonconvex relaxation from the existing conditions for SSR.

**Theorem 6.** Let \( A = [A_1, A_2] \in \mathbb{R}^{M \times N} \) with \( A_1 \in \mathbb{R}^{M \times (N-r)} \) and \( A_2 \in \mathbb{R}^{M \times r} \). Suppose that \( A_2 \) is full column rank, and let \( A_1 x + A_2 y = b \) with \( \|x\|_0 = k \). For \( 0 < p \leq 1 \) and \( a > 1 \), if
\[
\delta_{(a+1)k} < \frac{a^{1-p/2} - 1}{a^{1-p/2} + 1},
\]
(25)

then \((x, y)\) is the unique minimizer of problem (6). Specifically, for all \( 0 < p \leq 1 \), if
\[
\delta_{(a+1)k} < \frac{\sqrt{a} - 1}{\sqrt{a} + 1},
\]
(26)

then \((x, y)\) is the unique minimizer of problem (6).

**Proof.** Applying Theorem 5, here we only need to show that if (25) holds, we can find \( a_1 \) and \( a_2 \), such that (14) holds.

We consider the three cases in the following.

Case i \((a_1 \geq a_2 + 1)\). In this case, we easily obtain \( a_1 - a_2 \geq 1 \) and \( \delta_{a_1 k} \geq \delta_{(a+1)k} \). Therefore the following condition can guarantee the inequality (14):
\[
a_2 c_1 \delta_{a_1 k} + [(a_1 - a_2) c_2 + a_2] \delta_{a_1 k} < a_2 (c_1 + c_2 - 1) - a_1 c_2.
\]
(27)
Simplifying the previous inequality, we obtain
\[ \delta_{a,k} < \frac{a_2 c_1 - (a_1 - a_2) c_2 - a_2}{a_2 c_1 + (a_1 - a_2) c_2 + a_2} = \frac{a_1^{1-p/2} - (a_1 - a_2) a_2^{p/2} - 1}{a_1^{1-p/2} + (a_1 - a_2) a_2^{p/2} + 1}. \] (28)

In this case, employing \( a_2^{p/2} > 0 \), we easily get that \( a_1 = a_2 + 1 \) gives the maximum value of the right of the inequality (the strongest result) which satisfies the condition (14). That is,
\[ \delta_{[a+1],k} < \frac{(a_2 + 1)^{1-p/2} - a_2^{p/2} - 1}{(a_2 + 1)^{1-p/2} + a_2^{p/2} + 1}. \] (29)

**Case ii** (\( a_2 < a_1 < a_1 + 1 \)). In this case, we can get that \( 0 < a_1 - a_2 < 1 \) and \( \delta_{a,k} < \delta_{[a+1],k} \). Similarly, the following condition can guarantee the inequality (14):
\[ a_2 c_1 \delta_{[a+1],k} + [(a_1 - a_2) c_2 + a_2] \delta_{[a+1],k} < a_2 (c_1 + c_2 - 1) - a_1 c_2. \] (30)

Simplifying the previous inequality, we obtain
\[ \delta_{[a+1],k} < \frac{a_1^{1-p/2} - (a_1 - a_2) a_2^{p/2} - 1}{a_1^{1-p/2} + (a_1 - a_2) a_2^{p/2} + 1}. \] (31)

In this case, employing \( a_2^{p/2} > 0 \), we get that \( a_1 = a_2 \) gives the maximum value of the right of the inequality; that is,
\[ \delta_{[a+1],k} < \frac{a_2^{1-p/2} - 1}{a_2^{1-p/2} + 1}. \] (32)

**Case iii** (\( a_1 < a_2 \)). In this case, it is clear that \( 0 < a_2 - a_1, a_1 < a_2 + 1, \) and \( \delta_{a,k} < \delta_{[a+1],k} \). So the following condition can guarantee the inequality (14):
\[ a_2 c_1 \delta_{[a+1],k} + [(a_2 - a_1) c_2 + a_2] \delta_{[a+1],k} < a_2 (c_1 - c_2 - 1) + a_1 c_2. \] (33)

Simplifying the previous inequality, we obtain
\[ \delta_{[a+1],k} < \frac{a_1^{1-p/2} - (a_1 - a_2) a_2^{p/2} - 1}{a_1^{1-p/2} + (a_1 - a_2) a_2^{p/2} + 1}. \] (34)

In this case, employing \( a_2^{p/2} > 0 \), we chose \( a_2 - a_1 \to 0 \) to give the maximum value of the right of the inequality. That is,
\[ \delta_{[a+1],k} < \frac{a_2^{1-p/2} - 1}{a_2^{1-p/2} + 1}. \] (35)

It is easy to see that \( (a_2 + 1)^{1-p/2} - a_2^{p/2} - 1) / (a_2 + 1)^{1-p/2} + a_2^{p/2} + 1 \) is \( (a_2 - 1)^{1-p/2} - 1 / (a_2 - 1)^{1-p/2} + 1 \). In fact, \( (a_2 + 1)^{1-p/2} + a_2^{p/2} + 1 > a_2^{p/2} + 1 \). On the other hand, \( (a_2 + 1)^{1-p/2} - a_2^{p/2} - 1 = a_2 + 1 - (a_2^{p/2} - 1) = a_2 + 1 - (a_2^{p/2} - 1) = a_2 + 1 - (a_2^{p/2} - 1) < 0, \) which means \( (a_2 + 1)^{1-p/2} - a_2^{p/2} - 1 < a_2^{1-p/2} - 1. \)

**Figure 1:** The upper bound of \( p \)-RIC of order \((a+1)k\) with particular values of \( p \).

Therefore, combining the previous three cases, we obtain that one can choose \( a_1 = a_2 \) to get the weakest sufficient condition. It is easy to see that \( a_1 = a_2 \) satisfying the assumptions of Theorem 5.

After the previous discussion, using condition (25) and choosing \( a_1 = a_2 = a \), we can derive condition (14).

Specifically, we consider the following function:
\[ f(p) = \frac{a_1^{1-p/2} - 1}{a_1^{1-p/2} + 1} = 1 - \frac{2}{a_1^{1-p/2} + 1}. \] (36)

Clearly,
\[ f'(p) = - \frac{a_1^{1-p/2} \ln a}{(a_1^{1-p/2} + 1)^2} < 0, \] (37)

and hence \( f(p) \) is a decreasing function of \( p \). Thus, for all \( 0 < p \leq 1 \), condition
\[ \delta_{(a+1),k} < \frac{\sqrt{a} - 1}{\sqrt{a} + 1} \] (38)
can guarantee condition (14).

The proof is completed.

**4. Final Remark**

In this paper, we studied the restricted \( p \)-isometry property to the partially sparse signal recovery problem and proposed a sufficient \( p \)-RIP condition for exactly recovering partially sparse signal via the partially \( l_p \)-minimization problem with \( p \in (0,1) \). It is worth generalizing the \( p \)-RIP condition.
from the vector case to the matrix case. Note that the well-known low-rank matrix recovery (LMR) problem has many applications and appeared in the literature of a diverse set of fields including matrix completion, quantum state tomography, face recognition, magnetic resonance imaging (MRI), computer vision, and system identification and control; see, for example, [21, 22] for more details and the reference therein. In particular, Oymak et al. [21] showed that several sufficient RIP recovery conditions for $k$ sparse vector are also sufficient for recovery of matrices of rank up to $2k$ via Schatten $p$-norm minimization. According to our approach to extend the $p$-RIP bound from SSP to partially SSR, we can obtain some different restricted $p$-isometry properties for LMR problem by using the idea in [21].

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