Research Article

The Solutions of Mixed Monotone Fredholm-Type Integral Equations in Banach Spaces

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By introducing new definitions of $\phi$ convex and $-\phi$ concave quasioperator and $V_0$ quasilower and $u_0$ quasiupper, by means of the monotone iterative techniques without any compactness conditions, we obtain the iterative unique solution of nonlinear mixed monotone Fredholm-type integral equations in Banach spaces. Our results are even new to $\phi$ convex and $-\phi$ concave quasioperator, and then we apply these results to the two-point boundary value problem of second-order nonlinear ordinary differential equations in the ordered Banach spaces.

1. Introduction

In this paper, we will consider the following nonlinear Fredholm integral equation:

$$u(t) = \int_I H(t, s, u(s)) \, ds, \quad t \in I,$$

where $I = [a, b]$ and $H \in C[I \times I \times E, E]$, $E$ is a real Banach space with the norm $\| \cdot \|$, and there exists a function $G \in C[I \times I \times E \times E]$ such that for any $(t, s, x) \in I \times I \times E$

$$H(t, s, x) = G(t, s, x, x).$$

By introducing new definitions of $\phi$ convex and $-\phi$ concave quasioperator and $V_0$ quasilower and $u_0$ quasiupper, by means of the monotone iterative techniques without any compactness conditions which are of the essence in [2–4, 7, 8, 14], we obtain the iterative unique solution of nonlinear mixed monotone Fredholm-type integral equations in Banach spaces and then apply these results to the two-point boundary value problem of second-order nonlinear ordinary differential equations.

2. Preliminaries and Definitions

Let $P$ be a cone in $E$, that is, a closed convex subset such that $\lambda P \subset P$ for any $\lambda \geq 0$ and $P \cap \{-P\} = \{0\}$. By means of $P$, a partial order $\leq$ is defined as $x \leq y$ if and only if $y - x \in P$. A cone $P$ is said to be normal if there exists a constant $N > 0$ such that $x, y \in E, \theta \leq x \leq y$ implies $\|x\| \leq N \|y\|$, where $\theta$ denotes the zero element of $E$ (see [2, 14]), and we call the smallest number $N$ the normal constant of $P$ and denote $N_P$. The cone $P$ is normal if and only if every ordered interval $[x, y] = \{z \in E : x \leq z \leq y\}$ is bounded.

Let $P_I = \{u \in C[I, E] : u(t) \geq \theta \text{ for all } t \in I\}$, where $C[I, E]$ denotes the Banach space of all the continuous
mapping \( u : I \rightarrow E \) with the norm \( \| u \|_C = \max_{t \in I} |u(t)| \).
It is clear that \( P_I \) is a cone of space \( C[I, E] \), and so it defines a partial ordering in \( C[I, E] \). Obviously, the normality of \( P \) implies the normality of \( P_I \) and the normal constants of \( P_I \), and \( P \) are the same.

Let \( u_0, v_0 \in C[I, E] \). Then, \( u_0, v_0 \) are said to be coupled lower and upper quasi-solutions of (1) if

\[
\begin{align*}
    u_0(t) &\leq \int_I G(t, s, u_0(s), v_0(s)) \, ds, \quad t \in I, \\
v_0(t) &\geq \int_I G(t, s, v_0(s), u_0(s)) \, ds, \quad t \in I.
\end{align*}
\]

If the equality in (3) holds, then \( u_0, v_0 \) are said to be coupled quasi-solutions of (1).

We will always assume in this paper that \( P \) is a normal cone of \( E \). For any \( u_0, v_0 \in C[I, E] \) such that \( v_0 \leq w_0 \), we define the ordered interval \( D = [u_0, v_0] = \{ u \in C[I, E] : u_0 \leq u \leq v_0 \} \).

Next, we will give the new definition of \( \phi \) convex and \( \varphi \) concave quasi operator and \( v_0 \) quasi-lower and \( u_0 \) quasi-upper.

**Definition 1.** Suppose that, \( G \in C[I \times I \times E \times E, E] \). Then \( G \) is called \( \phi \) convex and \( \varphi \) concave quasi operator, if there exist functions

\[
\begin{align*}
    \phi : (0, \infty) \times (0, \infty) &\rightarrow (0, \infty), \\
    \varphi : (0, \infty) \times (0, \infty) &\rightarrow (0, \infty),
\end{align*}
\]

such that

\[
\begin{align*}
    (1) \quad &G(t, s, \alpha u, \beta v) \geq \phi(\alpha, \beta)G(t, s, u, v), \quad \alpha < \beta, \quad \alpha, \beta \in (0, \infty), \text{ for all } u, v \in E, \\
    (2) \quad &G(t, s, \alpha u, \beta v) \leq \varphi(\alpha, \beta)G(t, s, u, v), \quad \alpha \geq \beta, \quad \alpha, \beta \in (0, \infty), \text{ for all } u, v \in E.
\end{align*}
\]

**Definition 2.** Suppose that \( G \in C[I \times I \times E \times E, E] \). \( u_0 \in P \). Then, \( G \) is called \( u_0 \) quasi-upper, if for any \( u, v \in E \), \( u, v < u_0 \) such that \( \int_I G(t, s, u, v) \, ds < u_0 \).

**Definition 3.** Suppose that \( G \in C[I \times I \times E \times E, E] \). \( v_0 \in E \). Then, \( G \) is called \( v_0 \) quasi-lower, if for any \( u, v \in E \), \( u, v > v_0 \) such that \( \int_I G(t, s, u, v) \, ds > v_0 \).

Let us list the following assumption for convenience.

**(H1)** \( G \) is uniformly continuous on \( I \times I \times E \times E \), and \( G \) is \( \phi \) convex and \( \varphi \) concave quasi operator.

**(H2)** \( G(t, s, x, y) \) is nondecreasing in \( x \in E \) for fixed \( (t, s, y) \in I \times I \times E \). \( G(t, s, x, y) \) is nonincreasing in \( y \in E \) for fixed \( (t, s, x) \in I \times I \times E \).

**(H3)** \( \phi(\alpha, \beta), \varphi(\alpha, \beta) \) are all increasing in \( \alpha \), decreasing in \( \beta \), and \( \phi(\alpha_0, \beta_0) \geq \alpha_0, \varphi(\beta_0, \alpha_0) \leq \beta_0 \) if and for \( \alpha, \beta \in [\alpha_0, \beta_0], \alpha < \beta \),

\[
\begin{align*}
    \varphi(\beta, \alpha) &- \phi(\alpha, \beta) \leq l(\beta - \alpha), \quad 0 < l < 1.
\end{align*}
\]

**3. The Main Result**

The main results of this paper are the following three theorems.

**Theorem 4.** Let \( P \) be a normal cone of \( E \), let \( u_0, v_0 \in P \) be coupled lower and upper quasi-solutions of (1). Assume that conditions (H1), (H2), and (H3) hold and

**(H2)** There exists \( w_0 \in P \) such that \( u_0 \leq w_0 \leq v_0 \), and for \( \alpha_0, \beta_0 \in (0, \infty) \) of (H3) such that \( u_0 \geq \alpha_0 w_0, \beta_0 w_0 \geq v_0 \).

Then, (1) has a unique solution \( x^*(t) \in D = [u_0, v_0] \), and for any initial \( x_0, y_0 \in [u_0, v_0] \), one has

\[
\begin{align*}
    x_n(t) &\rightarrow x^*(t), \quad y_n(t) \rightarrow x^*(t), \\
    \text{uniformly on } t \in I \quad \text{as } n \rightarrow \infty,
\end{align*}
\]

where \( \{x_n(t)\}, \{y_n(t)\} \) are defined as

\[
\begin{align*}
    x_n(t) &\doteq \int_I G(t, s, x_{n-1}(s), y_{n-1}(s)) \, ds, \\
    y_n(t) &\doteq \int_I G(t, s, y_{n-1}(s), x_{n-1}(s)) \, ds, \quad t \in I.
\end{align*}
\]

**Proof.** We first define the operator \( A : [u_0, v_0] \times [u_0, v_0] \rightarrow C[I, E] \) by the formula

\[
A(u, v) = \int_I G(t, s, u(s), v(s)) \, ds.
\]

It follows from the assumption (H2) that \( A \) is a mixed monotone operator, that is, \( A(u, v) \) is nondecreasing in \( u \in [u_0, v_0] \) and nonincreasing in \( v \in [u_0, v_0] \), and \( u_0 \leq A(u_0, v_0), A(v_0, u_0) \leq v_0 \).

By (7), we have \( x_n(t) = A(x_{n-1}(t), y_{n-1}(t)), y_n(t) = A(y_{n-1}(t), x_{n-1}(t)) \) and set \( w_n(t) = A(w_{n-1}(t), w_{n-1}(t)) \) for initial \( w_0 \) in (H2), and we also define that

\[
\begin{align*}
    u_n(t) &= A(u_{n-1}(t), v_{n-1}(t)), \\
    v_n(t) &= A(v_{n-1}(t), u_{n-1}(t)).
\end{align*}
\]

Since \( A \) is a mixed monotone operator, it is easy to see that

\[
\begin{align*}
    u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \\
    u_n \leq w_n \leq v_n.
\end{align*}
\]

Obviously, by induction, it is easy to see that

\[
\begin{align*}
    u_n &\geq \alpha_n w_n, \quad v_n \leq \beta_n w_n, \quad n = 0, 1, \ldots, \\
    \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq 1 \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0, \\
    \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq 1 \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0,
\end{align*}
\]

where \( \alpha_n = \phi(\alpha_{n-1}, \beta_{n-1}), \beta_n = \phi(\beta_{n-1}, \alpha_{n-1}), n = 1, 2, \ldots \).

In fact, by the assumption (H3), we have that inequality (11) holds as \( n = 0 \). Suppose that inequality (11) holds as \( n = k, \)
that is, \( u_k \geq \alpha_k w_k, v_k \leq \beta_k w_k \). Then, as \( n = k + 1 \), by the assumption \((H_3)\), we have

\[
u_{k+1} = A(u_k, v_k) = \int_I G(t, s, u_k(s), v_k(s)) \, ds
\]

\[
\geq \int_I G(t, s, \alpha_k w_k, \beta_k w_k) \, ds
\]

\[
\geq \phi(\alpha_k, \beta_k) \int_I G(t, s, w_k(s), w_k(s)) \, ds = \alpha_{k+1} w_{k+1},
\]

\[
u_{k+1} = A(u_k, v_k) = \int_I G(t, s, v_k(s), u_k(s)) \, ds
\]

\[
\leq \int_I G(t, s, \beta_k w_k, \alpha_k w_k) \, ds
\]

\[
\leq \varphi(\beta_k, \alpha_k) \int_I G(t, s, w_k(s), w_k(s)) \, ds = \beta_{k+1} w_{k+1},
\]

(13)

Then, it is easy to show by induction that inequality (11) holds.

For inequality (12), by \( u_{k+1} \leq w_{k+1} \leq v_{k+1} \) and the above discussion, we have \( 0 < \alpha_{k+1} \leq 1 \leq \beta_{k+1} \). Obviously, it follows from the assumption \((H_2)\) that \( \alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0 \). Suppose that \( \alpha_{k+1} \leq \alpha_k, \beta_k \leq \beta_{k+1} \), so it is easy to see by \((H_2)\) that

\[
\phi(\alpha_{k-1}, \beta_{k-1}) \leq \phi(\alpha_k, \beta_k),
\]

\[
\varphi(\beta_k, \alpha_k) \leq \varphi(\beta_{k-1}, \alpha_{k-1}),
\]

(14)

that is, \( \alpha_k \leq \alpha_{k+1}, \beta_k \leq \beta_{k+1} \). Then, it is easy to show by induction that inequality (12) holds.

Then, it follows from the inequality (12) that there exist limits of the sequences \( \{\alpha_k\}, \{\beta_k\} \). Suppose that there exist \( \alpha, \beta \) such that \( \alpha_n \to \alpha, \beta_n \to \beta, \) and \( n \to \infty \), and by \((H_3)\), we also have

\[
0 \leq \beta_n - \alpha_n = \varphi(\beta_n, \alpha_n) - \phi(\alpha_n, \beta_n) \leq l(\beta_n - \alpha_n) \leq \cdots \leq l^p(\beta_0 - \alpha_0),
\]

(15)

they \( 0 < l < 1 \), and taking limits in the above inequality as \( n \to \infty \), we have \( \alpha = \beta \).

Next, we will show that the sequences \( \{u_n\}, \{v_n\} \) are all Cauchy sequences on \( D \).

In fact, by (10) and (11), for any natural number \( p \), we know that

\[
\theta \leq u_{n+p} - u_n \leq \beta_n - \alpha_n \leq (\beta_n - \alpha_n) u_0,
\]

\[
\theta \leq v_n - v_{n+p} \leq \beta_n - \alpha_n \leq (\beta_n - \alpha_n) u_0.
\]

By the normality of \( P_I \), and (15), we have

\[
\|u_{n+p} - u_n\|_C \leq N_p l^p(\beta_0 - \alpha_0) \|u_0\|_C,
\]

\[
\|v_n - v_{n+p}\|_C \leq N_p l^p(\beta_0 - \alpha_0) \|u_0\|_C,
\]

(17)

where \( N_p \) is a normal constant. So \( \{u_n\}, \{v_n\} \) are all Cauchy sequences on \( D \), and then there exists \( u^*, v^* \in [u_0, v_0] \) such that \( \lim_{n \to \infty} u_n = u^* \), \( \lim_{n \to \infty} v_n = v^* \).

It is easy to know by (10) and (11) that

\[
\theta \leq v_n - u_n \leq \beta_n - \alpha_n \leq (\beta_n - \alpha_n) u_0 \leq l^n(\beta_0 - \alpha_0) u_0,
\]

(18)

so by the normality of \( P_I \), we have

\[
\|v_n - u_n\|_C \leq N_p l^n(\beta_0 - \alpha_0) \|u_0\|_C,
\]

(19)

and taking limits in the above inequality as \( n \to \infty \), we have \( x^* = u^* = v^* \in [u_0, v_0] \), and for any natural number \( n \), we also have \( u_n \leq x^* \leq v_n \), \( t \in I \).

Then, by the mixed monotone quality of \( A \) we have

\[
u_{n+1} = A(u_n, v_n) \leq A(x^*, x^*) \leq A(v_n, u_n) = v_{n+1},
\]

(20)

and taking limits in above inequality as \( n \to \infty \), we know that

\[
x^* = A(x^*, x^*),
\]

(21)

that is, \( x^* \in [u_0, v_0] \) is the fixed point of \( A \); thus, \( x^* \) is the solution of (1) on \( D = [u_0, v_0] \).

Furthermore, we will show that the solution is unique. Suppose that \( y^* \in [u_0, v_0] \) satisfy \( y^* = A(y^*, y^*) \). Then, by the mixed monotone quality of \( A \) and induction, for any natural number \( n \), it is easy to have that \( u_n \leq y^* \leq v_n \). Then, by the normality of \( P_I \) and taking limits in the above inequality as \( n \to \infty \) and the above discussion, we have \( y^* = x^* \).

For any initial \( x_0, y_0 \in [u_0, v_0] \), by (7) and (8), the mixed monotone quality of \( A \) and induction, for any natural number \( n \), we have \( u_n(t) \leq x_0(t) \leq y_0(t) \leq v_n(t), t \in I \). Then, the normality of \( P_I \) and (19) imply that

\[
\|y_n - v_n\|_C \leq N_p l^n(\beta_0 - \alpha_0) \|u_0\|_C
\]

(22)

Thus, the sequence \( \{x_n(t)\}, \{y_n(t)\} \) all converges uniformly to \( x^*(t) \) on \( t \in I \). This completes the proof of Theorem 4. \( \square \)

**Theorem 5.** Let \( P \) be a normal cone of \( E \), let \( u_0, v_0 \in P \) be coupled lower and upper quasi-solutions of (1). Assume that conditions \((H_1), (H_2), \) and \((H_3)\) hold.

\((H_4)\) \( G \) is \( u_0 \) quasi-upper, and there exists \( w_0 \in P \) such that \( w_0 < u_0 < v_0 \), and there exist \( \alpha_0 = \sup\{\alpha > 0 : u_0 \geq \alpha u_0\}, \beta_0 = \inf\{\beta > 0 : \beta \geq w_0\} \).

Then, (1) has a unique solution \( x^*(t) \in D = [u_0, v_0] \), and for any initial \( x_0, y_0 \in [u_0, v_0] \), one has

\[
x_n(t) \to x^*(t), \quad y_n(t) \to x^*(t),
\]

uniformly on \( t \in I \) as \( n \to \infty \),

where \( \{x_n(t)\}, \{y_n(t)\} \) are defined as

\[
x_n(t) = \int_I G(t, s, x_{n-1}(s), y_{n-1}(s)) \, ds,
\]

(23)

\[
y_n(t) = \int_I G(t, s, y_{n-1}(s), x_{n-1}(s)) \, ds,
\]

(24)

\( t \in I \).
Proof. We first define the operator $A : [u_0, \nu_0] \times [u_0, \nu_0] \to C[I, E]$ by the formula
\[
A(u, v) = \int_I G(t, s, u(s), v(s)) \, ds.
\]
(8')

It follows from the assumption $(H_2)$ that $A$ is a mixed monotone operator, that is, $A(u, v)$ is nondecreasing in $u \in [u_0, \nu_0]$ and nonincreasing in $v \in [u_0, \nu_0]$ and $u_0 \leq A(u_0, \nu_0), A(\nu_0, u_0) \leq \nu_0$. By (7), we have $x_n(t) = A(x_{n-1}(t), y_{n-1}(t)) = A(y_{n-1}(t), x_{n-1}(t))$ and set $w_n(t) = A(u_{n-1}(t), w_{n-1}(t))$, and we also define
\[
\begin{align*}
  u_n(t) &= A(u_{n-1}(t), v_{n-1}(t)), \\
  v_n(t) &= A(v_{n-1}(t), u_{n-1}(t)).
\end{align*}
\]
(25)

Since $A$ is a mixed monotone operator, it is easy to see that
\[
0 \leq u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_0. \tag{10'}
\]

Because $G$ is $u_0$ quasi-upper and $w_0 < u_0$, we have
\[
w_1(t) = A(u_0(t), w_0(t)) = \int_I G(t, s, u_0(s), w_0(s)) \, ds < u_0.
\]
(26)

So for any natural number $n$, by induction, we know that $w_n(t) = A(u_{n-1}(t), w_{n-1}(t)) < u_0$.

It is easy to see by induction that
\[
\begin{align*}
  u_k &\geq \alpha_k w_k, \\
  v_k &\leq \beta_k w_k, \\
  a_0 \leq a_1 \leq \cdots \leq a_k \leq \cdots \leq b_k \leq \cdots \leq b_1 \leq b_0, \tag{11'}
\end{align*}
\]
where $\alpha_k = \phi(\alpha_{k-1}, \beta_{k-1}), \beta_k = \phi(\beta_{k-1}, \alpha_{k-1}), k = 1, 2, \ldots$.

In fact, by the assumptions $(H_1)$ and $(H_3)$ and the above discussion, as $n = 0$, we have
\[
u_1 = A(u_0, v_0) = \int_I G(t, s, u_0(s), v_0(s)) \, ds
\]
\[
\geq \int_I G(t, s, \alpha_0 w_0, \beta_0 w_0) \, ds
\]
\[
\geq \phi(\alpha_0, \beta_0) \int_I G(t, s, w_0(s), v_0(s)) \, ds = \alpha_1 w_1,
\]
\[
v_1 = A(v_0, u_0) = \int_I G(t, s, v_0(s), u_0(s)) \, ds
\]
\[
\leq \int_I G(t, s, \beta_0 w_0, \alpha_0 w_0) \, ds
\]
\[
\leq \phi(\beta_0, \alpha_0) \int_I G(t, s, w_0(s), v_0(s)) \, ds = \beta_1 w_1.
\]

By the above two inequalities and assumption $(H_3)$, we have
\[
a_0 \leq a_1 = \phi(\alpha_0, \beta_0) \leq \phi(\beta_0, \alpha_0) = \beta_1 \leq b_0. \tag{29}
\]

Suppose that for $k - 1$ we have $u_{k-1} \geq \alpha_{k-1} w_{k-1}, \nu_{k-1} \leq \beta_{k-1} w_{k-1},$ and $\alpha_k \leq \alpha_{k-1} \leq \beta_{k-1} \leq \beta_k - 2$. Then, for $k + 1$, by the assumption $(H_3)$, we have
\[
u_k = A(u_{k-1}, v_{k-1}) = \int_I G(t, s, u_{k-1}(s), v_{k-1}(s)) \, ds
\]
\[
\geq \int_I G(t, s, \alpha_{k-1} w_{k-1}, \beta_{k-1} w_{k-1}) \, ds
\]
\[
\geq \phi(\alpha_{k-1}, \beta_{k-1}) \int_I G(t, s, w_{k-1}(s), v_{k-1}(s)) \, ds = \alpha_k w_k,
\]
\[
y_k = A(v_{k-1}, u_{k-1}) = \int_I G(t, s, v_{k-1}(s), u_{k-1}(s)) \, ds
\]
\[
\leq \int_I G(t, s, \beta_{k-1} w_{k-1}, \alpha_{k-1} w_{k-1}) \, ds
\]
\[
\leq \phi(\beta_{k-1}, \alpha_{k-1}) \int_I G(t, s, w_{k-1}(s), v_{k-1}(s)) \, ds = \beta_k w_k.
\]

By the above two inequalities and assumption $(H_3)$, we have
\[
\alpha_{k-1} = \phi(\alpha_{k-2}, \beta_{k-2}) \leq \phi(\alpha_{k-1}, \beta_{k-1}) = \alpha_k \leq \beta_k
\]
\[
= \phi(\beta_{k-1}, \alpha_{k-1}) \leq \phi(\beta_{k-2}, \alpha_{k-2}) = \beta_{k-1}. \tag{31}
\]

Then, it is easy to show by induction that inequalities (11') and (12') hold.

The following proof is similar to that of Theorem 4. This completes the proof of Theorem 5. □

By a similar argument to that of Theorem 5, we obtain the following results.

**Theorem 6.** Let $P$ be a normal cone of $E$, and let $u_0, \nu_0 \in P$ be coupled lower and upper quasi-solutions of (1). Assume that condition $(H_1), (H_2)$, and $(H_3)$ hold.

$(H'_1)$ $G$ is $\nu_0$ quasi-lower, and there exists $w_0 \in P$ such that $u_0 < v_0 < u_0$, and there exist $\alpha_0 = \sup\{\alpha > 0 : u_0 \geq \alpha w_0\}, \beta_0 = \inf\{\beta > 0 : v_0 \leq \beta w_0\}$.

Then, (1) has a unique solution $x^*(t) \in D = [u_0, \nu_0]$, and for any initial $x_0, y_0 \in [u_0, \nu_0]$, one has
\[
x_n(t) \longrightarrow x^*(t), \quad y_n(t) \longrightarrow x^*(t),
\]
uniformly on $t \in I$ as $n \longrightarrow \infty$.

where $\{x_n(t)\}, \{y_n(t)\}$ are defined as
\[
x_n(t) = \int_I G(t, s, x_{n-1}(s), y_{n-1}(s)) \, ds,
\]
\[
y_n(t) = \int_I G(t, s, y_{n-1}(s), x_{n-1}(s)) \, ds, \quad t \in I.
\]
Consider the following two-point BVP in the Banach space:

\[-u'' = f(t, u), \quad t \in J = [0, 1], \quad u(0) = u(1) = 0,\]  

where \( f \in C[J \times P \times P, P] \), \( P \) is a cone in a real Banach space \( E \). Suppose that there exists a mapping \( g \in C[J \times P \times P, P] \) such that \( f(t, x) = g(t, x, x) \), and that \( g \) satisfies the following conditions:

\( (C_1) \) \( g \) is uniformly continuous on \( J \times P \times P \) and \( G \) is \( \phi \)-concave quasi operator,

\( (C_2) \) \( g(t, x, y) \) is nondecreasing in \( x \in P \) for fixed \( (t, y) \in J \times P \), and \( g(t, x, y) \) is nonincreasing in \( y \in P \), for fixed \( (t, x) \in J \times P \),

\( (C_3) \) there exist the bounded nonnegative Lebesgue integrable functions \( a(t), b(t), c(t), \) and \( d(t) \) satisfying

\[ \int_J a(s)ds < 8, \int_J c(s)ds < 8 \]

such that

\[ a(t)x + b(t) \leq g(t, x, y) \leq c(t)x + d(t), \quad t \in J, x, y \in P. \]  

\[ (35) \]

It is well known that \( u \in C^2[J, P] \) is a solution of BVP (34) in \( C^2[J, P] \) if and only if \( u \in C[J, P] \) is a solution of the following integral equation:

\[ u(t) = \int_J h(t, s)g(s, u(s), u(s))ds, \quad t \in J, \]  

where

\[ h(t, s) = \begin{cases} t(1-s), & t \leq s, \\ s(1-t), & t > s. \end{cases} \]  

\[ (37) \]

**Lemma 7.** If assumption \( (C_3) \) holds, then there exists \( u_0, v_0 \in C[J, P] \) such that

\[ u_0(t) \leq \int_J h(t, s)g(s, u_0(s), v_0(s))ds, \quad t \in J, \]  

\[ v_0(t) \geq \int_J h(t, s)g(s, v_0(s), u_0(s))ds, \quad t \in J. \]  

\[ (38) \]

**Proof.** In fact, let

\[ L_1u(t) = \int_J h(t, s)a(s)u(s)ds, \]

\[ x_0(t) = \int_J h(t, s)b(s)ds, \quad t \in J, \]

\[ L_2v(t) = \int_J h(t, s)c(s)v(s)ds, \]

\[ y_0(t) = \int_J h(t, s)d(s)ds, \quad t \in J. \]  

\[ (39) \]

Obviously, by assumption \( (C_3) \), we can get that \( \| L_1 \| \leq \max_{t \in J} ((1 - t)/2) \int_J a(s)ds = (1/8) \int_J a(s)ds < 1 \), then the equation \( (I - L_1)u = x_0 \) has a unique solution

\[ u_0(t) = (I - L_1)^{-1}x_0 = \sum_{n=0}^{\infty} L_1^n x_0 \in P_J. \]  

\[ (40) \]

Similarly, the equation \( (I - L_2)v = y_0 \) has a unique solution

\[ v_0(t) = (I - L_2)^{-1}y_0 = \sum_{n=0}^{\infty} L_2^n y_0 \in P_J. \]  

\[ (41) \]

Thus, by assumption \( (C_3) \), for any \( t \in J \), we have

\[ \int_J h(t, s)g(s, u_0(s), v_0(s))ds \]

\[ \geq \int_J h(t, s)(a(s)x + b(s))ds \]

\[ = L_1u_0(t) + x_0(t) = u_0(t), \]

\[ (42) \]

\[ \int_J h(t, s)g(s, v_0(s), u_0(s))ds \]

\[ \leq \int_J h(t, s)(c(s)v_0(s) + b(s))ds \]

\[ = L_2v_0(t) + y_0(t) = v_0(t), \]

that is, \( (38) \) holds. \qed

**Theorem 8.** Let \( P \) be a normal cone of \( E \). Assume that \( (C_1) \) and \( (C_3) \) hold,

\( (C_4) \) there exists \( w_0 \in P_1 \) and \( u_0, v_0 \) in \( (38) \) of Lemma 7 such that \( u_0 < w_0 < v_0 \), and also there exists \( \alpha_0, \beta_0 \in (0, \infty) \) such that \( u_0 \geq \alpha_0 w_0, \beta_0 \geq \alpha_0 \),

\( (C_5) \) \( \phi(\alpha, \beta) \), \( \psi(\alpha, \beta) \) are all increasing in \( \alpha \), decreasing in \( \beta \) and \( \phi(\alpha_0, \beta_0) \geq \alpha_0, \psi(\beta_0, \alpha_0) \leq \beta_0 \), for \( \alpha, \beta \in [\alpha_0, \beta_0], \alpha < \beta, \)

\[ \phi(\beta, \alpha) - \phi(\alpha, \beta) \leq l(\beta - \alpha), \quad 0 < l < 1. \]  

\[ (43) \]

Then, \( (34) \) has a unique solution \( x^*(t) \in D = [u_0, v_0], \) and for any initial \( x_0, y_0 \in [u_0, v_0], \) one has

\[ x_n(t) \to x^*(t), \quad y_n(t) \to x^*(t), \]

uniformaly on \( t \in I \) as \( n \to \infty \),

\[ (44) \]

where \( \{x_n(t)\}, \{y_n(t)\} \) are defined as

\[ x_n(t) = \int_J h(t, s)g(s, x_{n-1}(s), y_{n-1}(s))ds, \]

\[ y_n(t) = \int_J h(t, s)g(s, y_{n-1}(s), x_{n-1}(s))ds, \quad t \in J. \]  

\[ (45) \]
Proof. It is easy to see by conditions (C_1) and (C_2) that \( G(t, s, x, y) = h(t, s)g(s, x, y) \) satisfy the conditions (H_1) and (H_2) of Theorem 4. By (C_3) and (38), we have \( u_0, v_0 \in C[I, P] \) as coupled lower and upper quasi-solutions of (34).

Thus, the assumption (H_1)–(H_2) of Theorem 4 is satisfied from the assumption (C_1)–(C_5) of Theorem 8. The conclusion of Theorem 8 follows from Theorem 4.

Example 9. In fact, we can construct the function \( f(t, x) \) in Theorem 8.

Let

\[
f(t, x) = g(t, x, y) = x + \frac{1}{y}, \quad t \in [0, 1],
\]

\[
\phi(\alpha, \beta) = \sin \alpha + \frac{1}{2\beta},
\]

\[
\alpha \in \left[0, \frac{\pi}{2}\right],
\]

\[
\varphi(\alpha, \beta) = 3\alpha - 5\beta,
\]

then

\[
G(t, s, \alpha x, \beta y) = h(t, s) g(s, \alpha x, \beta y)
\]

\[
= h(t, s) \left( \alpha x + \frac{1}{\beta y} \right)
\]

\[
\geq h(t, s) \left( \sin \alpha + \frac{1}{2\beta} \right) \left( x + \frac{1}{y} \right)
\]

\[
= \phi(\alpha, \beta) G(t, s, x, y),
\]

\[
G(t, s, \alpha x, \beta y) = h(t, s) g(s, \alpha x, \beta y)
\]

\[
= h(t, s) \left( \alpha x + \frac{1}{\beta y} \right)
\]

\[
\leq h(t, s) (3\alpha - 5\beta) \left( x + \frac{1}{y} \right)
\]

\[
= \varphi(\alpha, \beta) G(t, s, x, y).
\]

Thus, \( G \) is \( \phi \) convex and \( -\varphi \) concave quasi operator and thus satisfies (C_1).

It is easy to check that \( g(t, x, y) \) is nondecreasing in \( x \) for fixed \( t, y \) and is nonincreasing in \( y \) for fixed \( t, x \) and thus satisfies (C_2).

There exist \( a(t) = t/2, b(t) = t/100, c(t) = 2t \), and \( d(t) = 100t \) satisfying

\[
\int_0^1 a(s) ds = \frac{1}{2} \int_0^1 t dt = \frac{1}{4} < 8,
\]

\[
\int_0^1 b(s) ds = 2 \int_0^1 s ds = 1 < 8,
\]

such that

\[
a(t) x + b(t) \leq g(t, x, y) \leq c(t) x + d(t).
\]

Thus, (C_3) holds.

There exist

\[
u_0 = \int_0^1 h(t, s) \left[ u_0(s) + \frac{1}{v_0(s)} \right] ds,
\]

\[
v_0 = 2u_0 = \int_0^1 h(t, s) \left[ \frac{1}{u_0(s)} \right] ds.
\]

Choose \( u_0 = (3/2)u_0 \) such that \( u_0 < u_0 < v_0 \), and also there exist \( \alpha_0 = 2/3, \beta_0 = 4/3 \) such that

\[
u_0 = \alpha_0 \frac{3}{2} u_0 = \alpha_0 u_0, \quad \beta_0 \frac{3}{2} u_0 = 2 u_0 = v_0.
\]

Thus, \( (C_4) \) is satisfied.

\[
\phi(\alpha, \beta), \varphi(\alpha, \beta) \text{ are all increasing in } \alpha \text{ and nondecreasing in } \beta,
\]

\[
\phi \left( \frac{2}{3}, \frac{4}{3} \right) = \sin \frac{2}{3} + \frac{1}{2 \times (4/3)} = \frac{2}{3} = \alpha_0,
\]

\[
\varphi \left( \frac{4}{3}, \frac{2}{3} \right) = 3 \times \frac{4}{3} - 5 \times \frac{2}{3} = \frac{2}{3} \leq \frac{4}{3} = \beta_0,
\]

and for \( \alpha, \beta \in [2/3, 4/3], \alpha < \beta \), we have

\[
\varphi(\beta, \alpha) - \phi(\alpha, \beta) = 3\beta - 5\alpha - \sin \alpha - \frac{1}{2\beta} \leq \frac{99}{100} (\beta - \alpha).
\]

Thus, \( (C_5) \) also holds.

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References


