Research Article

Nearly Quadratic $n$-Derivations on Non-Archimedean Banach Algebras

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Let $n > 1$ be an integer, let $A$ be an algebra, and $X$ be an $A$-module. A quadratic function $D : A \to X$ is called a quadratic $n$-derivation if

$$D(\prod_{i=1}^{n}a_i) = D(a_1) a_2 \cdots a_n + a_1 D(a_2) a_3 \cdots a_n + \cdots + a_1 a_2 \cdots a_{n-1} D(a_n)$$

for all $a_1, \ldots, a_n \in A$. We investigate the Hyers-Ulam stability of quadratic $n$-derivations from non-Archimedean Banach algebras into non-Archimedean Banach modules by using the Banach fixed point theorem.

1. Introduction

A functional equation $(\xi)$ is stable if any function $g$ satisfying the equation $(\xi)$ approximately is near to a true solution of $(\xi)$.

The stability of functional equations was first introduced by Ulam [1] in 1964. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1978, Th. M. Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$, $(\epsilon > 0, p \in [0, 1])$. In 1994, a generalization of Th. M. Rassias theorem was obtained by Gavruta [4], who replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$ (see also [5–7]).

Every solution of the following functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.1)$$
is said to be a quadratic function [8]. It is well known that a mapping $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping $B_1$ such that $f(x) = B_1(x, x)$ for all $x$. The biadditive mapping $B_1$ is given by

$$B_1(x, y) = (1/4)(f(x + y) - f(x - y)).$$

The stability problem of the quadratic functional equation was proved by Skof [9] for mappings $f : A \rightarrow B$, where $A$ is a normed space and $B$ is a Banach space (see also [10, 11]). Let $A$ be an algebra and let $X$ be a $A$-bimodule. A quadratic function $D : A \rightarrow X$ is called a quadratic $n$-derivation if

$$D \left( \prod_{i=1}^{n} a_i \right) = D(a_1)a_2^2 \cdots a_n^2 + a_1^2D(a_2)a_3^2 \cdots a_n^2 + \cdots + a_1^2a_2^2 \cdots a_{n-1}^2D(a_n) \quad (1.2)$$

for all $a_1, \ldots, a_n \in A$. Recently, Gordji and Ghobadipour [12] introduced the quadratic derivations on Banach algebras. Indeed, they investigated the Hyers-Ulam-Aoki-Rassias stability and Ulam-Gavruta-Rassias type stability of quadratic derivations on Banach algebras.

More recently, Gordji et al. [13] investigated the Hyers-Ulam stability and the superstability of higher ring derivations on non-Archimedean Banach algebras (see also [12–32]). In this paper we investigate the Hyers-Ulam stability of quadratic $n$-derivations from non-Archimedean Banach algebras into non-Archimedean Banach modules by using the weighted space method (see [33]).

2. Preliminaries

Let us recall that a non-Archimedean field is a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial (see [34]).

**Definition 2.1.** Let $X$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

\begin{align*}
(\text{NA}_1) & \quad \|x\| = 0 \text{ if and only if } x = 0; \\
(\text{NA}_2) & \quad \|rx\| = |r|\|x\| \text{ for all } r \in \mathbb{K} \text{ and } x \in X; \\
(\text{NA}_3) & \quad \|x + y\| \leq \max\{\|x\|, \|y\|\} \text{ for all } x, y \in X \text{ (the strong triangle inequality).}
\end{align*}

In 1897, Hensel [35] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications. The most important examples of non-Archimedean spaces are $p$-adic numbers. Let $p$ be a prime number. For any nonzero rational number $x = (a/b)p^{n_2}$ such that $a$ and $b$ are integers not divisible by $p$, define the $p$-adic absolute value $|x|_p := p^{-n_2}$. Then $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$, which is called the $p$-adic number field.
Theorem 2.3. Let $X$ be a nonempty set and let $d : X \times X \to [0, \infty)$ satisfy the following properties:
\begin{enumerate}
  \item[(D$_1$)] $d(x, y) = 0$ if and only if $x = y$,
  \item[(D$_2$)] $d(x, y) = d(y, x)$ (symmetry),
  \item[(D$_3$)] $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (strong triangle inequality),
\end{enumerate}

for all $x, y, z \in X$. Then $(X, d)$ is called a non-Archimedean metric space. $(X, d)$ is called a non-Archimedean complete metric space if every $d$-Cauchy sequence in $X$ is $d$-convergent.

**Theorem 2.3 (Non-Archimedean Banach Contraction Principle).** Let $(X, d)$ be a non-Archimedean complete metric space and let $T : X \to X$ be a contraction; that is, there exists $a \in [0, 1)$ such that
\begin{equation}
  d(Tx, Ty) \leq ad(x, y), \quad \forall x, y \in X.
\end{equation}

Then there exists a unique element $a \in X$ such that $Ta = a$. Moreover, $a = \lim_{n \to \infty} T^n x$, and
\begin{equation}
  d(a, x) \leq d(x, Tx), \quad \forall x \in X.
\end{equation}

**Proof.** A similar argument as Archimedean case can be applied to show that $T$ has a unique element $a \in X$ such that $Ta = a$ and $a = \lim_{n \to \infty} T^n x$. It follows from strong triangle inequality that for all $x \in X$ and for each $n \in \mathbb{N}$, we have
\begin{equation}
  d(T^n x, x) \leq \max\{d(T(x), x), \ldots, d(T^n(x), T^{n-1}(x))\}
  \leq \max\{d(T(x), x), \ldots, a^{n-1}d(T(x), x)\}
  = d(T(x), x).
\end{equation}

\[ \blacksquare \]

3. Main Results

In this section $A$ denotes a non-Archimedean Banach algebra over a non-Archimedean field $\mathbb{K}$ and $X$ is a non-Archimedean Banach $A$-module.

**Theorem 3.1.** Let $\varphi : A \times A \to [0, \infty)$, $\varphi : A \times \cdots \times A \to [0, \infty)$ be functions. Let $f : A \to X$ be a given mapping such that $f(0) = 0$,
\begin{equation}
  \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)
\end{equation}

and that
\begin{equation}
  \|f\left(\prod_{i=1}^{n} x_i\right) - f(x_1)x_2^2 \cdots x_n^2 - x_1^2f(x_2)x_3^2 \cdots x_n^2 - \cdots - x_1^2 \cdots x_{n-1}^2f(x_n)\| \leq \varphi(x_1, \ldots, x_n)
\end{equation}
for all \(x_1, \ldots, x_n, x, y \in A\). Suppose that there exist a natural number \(k \in \mathbb{K}\) and \(L, K \in (0, 1)\), such that

\[
|k|^2 \varphi\left(k^{-1} x, k^{-1} y\right) \leq L \varphi(x, y), \quad |k|^2 \varphi\left(k^{-1} x_1, \ldots, k^{-1} x_n\right) \leq K \varphi(x_1, \ldots, x_n)
\]  

(3.3)

for all \(x_1, \ldots, x_n, x, y \in A\). Then there exists a unique quadratic \(n\)-derivation \(h\) from \(A\) into \(X\) such that

\[
\|f(x) - h(x)\| \leq \frac{L \Phi(x)}{|k|^2}
\]  

(3.4)

for all \(x \in A\), where

\[
\Phi(x) = \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi((k-1)x, x)\} \quad (x \in A).
\]  

(3.5)

Proof. By induction on \(i\), one can show that for all \(x \in A\) and \(i \geq 2\),

\[
\|f(ix) - i^2 f(x)\| \leq \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi((i-1)x, x)\}.
\]  

(3.6)

Let \(x = y\) in (3.1). Then

\[
\|f(2x) - 2^2 f(x)\| \leq \max\{\varphi(0, 0), \varphi(x, x)\} \quad (x \in A).
\]  

(3.7)

This proves (3.6) for \(i = 2\). Let (3.6) hold for \(i = 1, 2, \ldots, j\). Replacing \(x\) by \(jx\) and \(y\) by \(x\) in (3.1) for all \(x \in A\), we get

\[
\|f((j + 1)x) + f((j - 1)x) - 2f(jx) - 2f(x)\| \leq \max\{\varphi(0, 0), \varphi(jx, x)\}
\]  

(3.8)

for all \(x \in A\). Since

\[
f((j + 1)x) + f((j - 1)x) - 2f(jx) - 2f(x) = f((j + 1)x) - (j + 1)^2 f(x)
\]

\[
+ f((j - 1)x) - (j - 1)^2 f(x) - 2\left[f(jx) - j^2 f(x)\right]
\]  

(3.9)

for all \(x \in A\), it follows from induction hypothesis and (3.8) that for all \(x \in A\),

\[
\|f((j + 1)x) - (j + 1)^2 f(x)\| \leq \max\left\{\|f((j + 1)x) + f((j - 1)x) - 2f(jx) - 2f(x)\|ight.
\]

\[
, \|f((j - 1)x) - (j - 1)^2 f(x)\|, \|2\|f^2 f(x) - f(jx)\|\}
\]

\[
\leq \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi((j)x, x)\}.
\]  

(3.10)
This proves (3.6) for all $i \geq 2$. In particular
\[
\left\| f(kx) - k^2 f(x) \right\| \leq \Phi(x) \quad (x \in A). \tag{3.11}
\]
Replacing $x$ by $k^{-1}x$ in (3.11), we get
\[
\left\| f(x) - k^2 f(k^{-1}x) \right\| \leq \Phi(k^{-1}x) \leq \frac{L}{|k|^2} \Phi(x) \tag{3.12}
\]
for all $x \in A$. Let $\Omega$ be the set of all functions $u : A \to X$. We define the metric $d$ on $\Omega$ as follows:
\[
d(u, v) = \sup_{x \in A} D(x), \tag{3.13}
\]
where $D(x) = (\|u(x) - v(x)\|)/\Phi(x)$ if $\Phi(x) \neq 0$ and $D(x) = \|u(x) - v(x)\|$ if $\Phi(x) = 0$. One has the operator $J : \Omega \to \Omega$ by $J(u)(x) = k^2 u(k^{-1}x)$. Then $J$ is strictly contractive on $\Omega$, in fact, if
\[
\|u(x) - v(x)\| \leq \sigma \Phi(x) \quad (x \in A), \tag{3.14}
\]
then by (3.3),
\[
\|J(u)(x) - J(v)(x)\| = \|k^2 u(k^{-1}x) - v(k^{-1}x)\| \leq \sigma |k|^2 \Phi(k^{-1}x) \leq L \sigma \Phi(x), \quad (x \in A). \tag{3.15}
\]
It follows that
\[
d(J(u), J(v)) \leq Ld(u, v) \quad (u, v \in \Omega). \tag{3.16}
\]
Hence $J$ is a contractive with Lipschitz constant $L$. By Theorem 2.3, $J$ has a unique fixed point $h : A \to X$ and
\[
h(x) = \lim_{m \to \infty} J^m(f(x)) = \lim_{m \to \infty} k^{2m} f(k^{-m}x) \tag{3.17}
\]
for all $x \in A$.
Therefore
\[
\begin{align*}
\left\| h(x + y) + h(x - y) - 2h(x) - 2h(y) \right\| & = \lim_{m \to \infty} |k|^{2m} \left\| f(k^{-m}(x + y)) + f(k^{-m}(x - y)) - 2f(k^{-m}x) - 2f(k^{-m}y) \right\| \\
& \leq \lim_{m \to \infty} |k|^{2m} \varphi(k^{-m}x, k^{-m}y) \\
& \leq \lim_{m \to \infty} L^m \varphi(x, y) = 0
\end{align*} \tag{3.18}
\]
for all \( x, y \in A \). This shows that \( h \) is quadratic. It follows from Theorem 2.3 that

\[
d(f, h) \leq d(f(f), f),
\]

that is,

\[
\|f(x) - h(x)\| \leq \frac{L\Phi(x)}{|k|^2} (x \in A).
\]

Replacing \( x \) by \( k^{-m}x_i, i = 1, \ldots, n \) in (3.2), we get

\[
\|f \left( \prod_{i=1}^{n} k^{-mn}x_i \right) - f(k^{-m}x_1)k^{-2m(n-1)}x_2^2 \cdots x_n^2
\]

\[
- k^{-2m(n-1)}x_1^2f(k^{-m}x_2)x_3^2 \cdots k^{-2m(n-1)}x_n^2 - \cdots - x_1^2 \cdots x_{n-1}^2 f(k^{-m}x_n) \|
\]

\[
\leq \psi(k^{-m}x_1, \ldots, k^{-m}x_n),
\]

and so

\[
|k|^{2mn} \left\| f \left( \prod_{i=1}^{n} k^{-mn}x_i \right) - f(k^{-m}x_1)k^{-2m(n-1)}x_2^2 \cdots x_n^2
\]

\[
- k^{-2m(n-1)}x_1^2f(k^{-m}x_2)x_3^2 \cdots - k^{-2m(n-1)}x_n^2 \cdots x_{n-1}^2 f(k^{-m}x_n) \right\|
\]

\[
= 2^{2mn} f \left( \prod_{i=1}^{n} k^{-mn}x_i \right) - k^{2m} f(k^{-m}x_1)x_2^2 \cdots x_n^2
\]

\[
- x_1^2 k^{2m} f(k^{-m}x_2)x_3^2 \cdots - x_1^2 \cdots x_{n-1}^2 k^{2m} f(k^{-m}x_n) \right\|
\]

\[
\leq |k|^{2mn} \psi(k^{-m}x_1, \ldots, k^{-m}x_n) \leq |k|^{2mn} \frac{K^m}{|k|^{2m}} \psi(x_1, \ldots, x_n)
\]

for all \( x_1, \ldots, x_n \in A \) and each \( m \in \mathbb{N} \). By taking \( m \to \infty \), we have

\[
h \left( \prod_{i=1}^{n} x_i \right) = h(x_1)x_2^2 \cdots x_n^2 + x_1^2 h(x_2)x_3^2 \cdots x_n^2 + \cdots - x_1^2 \cdots x_{n-1}^2 h(x_n)
\]

for all \( x_1, \ldots, x_n \in A \).

In the following corollaries we will assume that \( A \) is a non-Archimedean Banach algebra over \( \mathbb{K} = \mathbb{Q}_p \) the field of \( p \)-adic numbers, where \( p > 2 \) is a prime number.
Corollary 3.2. Let \( r < 1 \) and let \( \varepsilon \) be \( \delta \) be positive real numbers. Suppose that \( f : A \to X \) is a mapping such that

\[
\| f(x + y) + f(x - y) - 2f(x) - 2(y) \| \leq \varepsilon \| x \|'' \| y \|^r, \\
\left\| f \left( \prod_{i=1}^{n} x_i \right) - f(x_1)x_2^2 \cdots x_n^2 - x_1^2f(x_2)x_3^2 \cdots x_n^2 - \cdots - x_1^2 \cdots x_{n-1}^2f(x_n) \right\| \\
\leq \delta \max\{\|x_1\|', \ldots, \|x_n\|'\}
\]

for all \( x_1, \ldots, x_n, x, y \in A \). Then there exists a unique quadratic \( n \)-derivation \( h \) from \( A \) into \( X \) such that

\[
\| f(x) - h(x) \| \leq \varepsilon p^{2r} \| x \|^{2r}
\]

for all \( x \in A \).

Proof. By (3.24), \( f(0) = 0 \). Let \( \varphi(x, y) = \varepsilon \| x \|'' \| y \|^r \) and \( \varphi(x_1, \ldots, x_n) = \delta \max\{\|x_1\|', \ldots, \|x_n\|'\} \) for all \( x_1, \ldots, x_n, x, y \in A \). Then

\[
[p]^{2} \varphi(p^{-1}x, p^{-1}y) = p^{2r-2} \varphi(x, y), \quad [p]^{2} \varphi(p^{-1}x_1, \ldots, p^{-1}x_n) = p^{r-2} \varphi(x_1, \ldots, x_n)
\]

for all \( x_1, \ldots, x_n, x, y \in A \).

Moreover,

\[
\Phi(x) = \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi((p - 1)x, x)\} = \varepsilon \| x \|^{2r} \quad (x \in A).
\]

Put \( L = p^{2r-2} \) and \( K = p^{r-2} \) in Theorem 3.1. Then there exists a unique quadratic \( n \)-derivation \( h \) from \( A \) into \( X \) such that

\[
\| f(x) - h(x) \| \leq \varepsilon p^{2r} \| x \|^{2r}
\]

for all \( x \in A \).\Box

Similarly, we can prove the following result.

Corollary 3.3. Let \( r < 2 \) and let \( \varepsilon \) be \( \delta \) be positive real numbers. Suppose that \( f : A \to X \) is a mapping such that

\[
\| f(x + y) + f(x - y) - 2f(x) - 2(y) \| \leq \varepsilon \max\{\|x\|'', \|y\|''\}, \\
\left\| f \left( \prod_{i=1}^{n} x_i \right) - f(x_1)x_2^2 \cdots x_n^2 - x_1^2f(x_2)x_3^2 \cdots x_n^2 - \cdots - x_1^2 \cdots x_{n-1}^2f(x_n) \right\| \\
\leq \delta \max\{\|x_1\|', \ldots, \|x_n\|'\}
\]

for all \( x_1, \ldots, x_n, x, y \in A \).
for all \( x_1, \ldots, x_n, x, y \in A \). Then there exists a unique quadratic \( n \)-derivation \( h \) from \( A \) into \( X \) such that

\[
\|f(x) - h(x)\| \leq \varepsilon p^n \|x\|^n
\]  

(3.30)

for all \( x \in A \).

**Remark 3.4.** We can use similar arguments to obtain corollaries like Corollaries 3.2 and 3.3, when \( r > 1 \) and \( r > 2 \).

By using the same technique of proving Theorem 3.1, we can prove the following result.

**Remark 3.5.** Let \( \varphi : A \times A \rightarrow [0, \infty) \), \( \psi : A \times \cdots \times A \rightarrow [0, \infty) \) be functions. Let \( f : A \rightarrow X \) be a given mapping such that \( f(0) = 0 \),

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)
\]  

(3.31)

and that

\[
\|f\left(\prod_{i=1}^{n} x_i\right) - f(x_1)x_2^2 \cdots x_2 \cdots x_{n}^2 - x_1^2 f(x_2)x_3^2 \cdots x_{n}^2 - \cdots - x_1^2 \cdots x_{n-1}^2 f(x_n)\| \leq \psi(x_1, \ldots, x_n)
\]  

(3.32)

for all \( x_1, \ldots, x_n, x, y \in A \). Suppose that there exist a natural number \( k \in \mathbb{N} \) and \( L, K \in (0, 1) \), such that

\[
\varphi(kx, y) \leq |k|^2 L \varphi(x, y), \quad \varphi(kx_1, \ldots, kx_n) \leq |k|^2 K \psi(x_1, \ldots, x_n)
\]  

(3.33)

for all \( x_1, \ldots, x_n, x, y \in A \). Then there exists a unique quadratic \( n \)-derivation \( d \) from \( A \) into \( X \) such that

\[
\|f(x) - d(x)\| \leq |k|^2 L \Phi(x)
\]  

(3.34)

for all \( x \in A \), where

\[
\Phi(x) = \max\left\{ \varphi(0, 0), \varphi(x, x), \varphi\left(\frac{x}{2}, x\right), \ldots, \varphi\left(\frac{x}{(k-1)}, x\right) \right\} \quad (x \in A).
\]  

(3.35)

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