Research Article

Nearly Derivations on Banach Algebras

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Let $n$ be a fixed integer greater than 3 and let $\lambda$ be a real number with $\lambda \neq (n^2 - n + 4)/2$. We investigate the Hyers-Ulam stability of derivations on Banach algebras related to the following generalized Cauchy functional inequality

$$\|\sum_{1 \leq i < j \leq n} f((x_i + x_j)/2 + \sum_{k=1}^{n-2} x_k) + f(\sum_{i=2}^{n} x_i) + f(x_i)\| \leq \|\lambda f(\sum_{i=1}^{n} x_i)\|.$$  

1. Introduction and Preliminaries

Let $\mathcal{A}$ be a Banach algebra and let $X$ be a Banach $\mathcal{A}$-bimodule. Then $X^*$, the dual space of $X$, is also a Banach $\mathcal{A}$-bimodule with module multiplications defined by

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \quad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle, \quad (a \in \mathcal{A}, \ x \in X, \ x^* \in X^*). \quad (1.1)$$

A bounded linear operator $D : \mathcal{A} \rightarrow X$ is called a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}). \quad (1.2)$$

Let $x \in X$. We define $\delta_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. $\delta_x$ is a derivation from $\mathcal{A}$ into $X$, which is called inner derivation. A Banach algebra $\mathcal{A}$ is amenable if every derivation from $\mathcal{A}$ into every dual $\mathcal{A}$-bimodule $X^*$ is inner. This definition was introduced by Johnson in [1]. A Banach algebra $\mathcal{A}$ is weakly amenable if every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner. Bade et al. [2] have introduced the concept of weak amenability for commutative Banach algebras.

A famous talk presented by Ulam in 1940 triggered the study of stability problems for various functional equations.

We are given a group \( G_1 \) and a metric group \( G_2 \) with metric \( \rho(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if \( f : G_1 \to G_2 \) satisfies \( \rho(f(xy), f(x)f(y)) < \delta \) for all \( x, y \in G_1 \), then a homomorphism \( h : G_1 \to G_2 \) exists with \( \rho(f(x), h(x)) < \epsilon \) for all \( x \in G_1 \)?

In the following year, Hyers was able to give a partial solution to Ulam’s question that was the first significant breakthrough and step toward more solutions in this area (see [5]). Since then, a large number of papers have been published in connection with various generalizations of Ulam’s problem and Hyers’ theorem.

Let \( n \) be a fixed integer greater than 3 and let \( \lambda \) be a real number with \( |\lambda| \neq (n^2 - n + 4)/2 \). We investigate the Hyers-Ulam stability of derivations on Banach algebras related to the following generalized Cauchy functional inequality:

\[
\left\| \sum_{1 \leq i < j \leq n} f\left( \frac{x_i + x_j}{2} + \sum_{k=1}^{n-2} x_k \right) + f\left( \sum_{i=2}^{n} x_i \right) + f(x_1) \right\| \leq \lambda \left\| f\left( \sum_{i=1}^{n} x_i \right) \right\|. \tag{1.3}
\]

## 2. Main Results

Let \( A \) be a Banach algebra and let \( X \) be a Banach \( A \)-module. From now on, the sum of \( f(x) \) and \( f(-x) \) will be denoted by \( \bar{f}(x) \). Also, \( f(ab) - f(a)b - af(b) \) will be denoted by \( \Delta f(a, b) \). In the following, we will use the Pascal formula:

\[
C(r, k) = C(r - 1, k) + C(r - 1, k - 1) \tag{2.1}
\]

here, \( C(r, k) \) denotes \( r! / k!(r - k)! \). Moreover, we assume that \( n_0 \in \mathbb{N} \) is a positive integer and suppose that \( \mathbb{T}_{1/n_0} := \{ e^{i\theta}; \ 0 \leq \theta \leq 2\pi/n_0 \} \).

**Lemma 2.1.** Let \( f : A \to X \) be a mapping such that

\[
\left\| \sum_{1 \leq i < j \leq n} f\left( \frac{x_i + x_j}{2} + \sum_{k=1}^{n-2} x_k \right) + f\left( \sum_{i=2}^{n} x_i \right) + f(x_1) \right\| \leq \lambda \left\| f\left( \sum_{i=1}^{n} x_i \right) \right\|. \tag{2.2}
\]

for all \( x_1, \ldots, x_n \in A \). Then \( f \) is Cauchy additive.

**Proof.** Substituting \( x_1, \ldots, x_n = 0 \) in the functional inequality (2.2), we get

\[
\| (C(n, 2) + 2)f(0) \| \leq \| \lambda f(0) \|. \tag{2.3}
\]
Since \( n \geq 3 \) and \( |\lambda| \neq (n^2 - n + 4)/2, f(0) = 0 \). Letting \( x_1 = x, x_2 = -x \) and \( x_3 = \cdots = x_n = 0 \) in (2.2) and using Pascal formula, we get

\[
\|(n - 2)f\left(\frac{x}{2}\right) + (C(n - 2, 2) + 1)f(0) + \tilde{f}(x)\| \leq \|\lambda f(0)\|,
\] (2.4)

for all \( x \in A \). Hence

\[
(n - 2)f\left(\frac{x}{2}\right) + \tilde{f}(x) = 0
\] (2.5)

for all \( x \in A \). Letting \( x_1 = 2x, x_2 = -x, x_3 = -x \) and \( x_4 = \cdots = x_n = 0 \) in (2.2), we get

\[
\|2f\left(\frac{-x}{2}\right) + (n - 3)f(-x) + f(x) + 2(n - 3)f\left(\frac{x}{2}\right) + C(n - 3, 2)f(0) + \tilde{f}(2x)\| \leq \|\lambda f(0)\|
\] (2.6)

for all \( x \in A \). Hence

\[
2f\left(\frac{-x}{2}\right) + (n - 3)f(-x) + f(x) + 2(n - 3)f\left(\frac{x}{2}\right) + \tilde{f}(2x) = 0,
\] (2.7)

\[
2f\left(\frac{x}{2}\right) + (n - 3)f(x) + f(-x) + 2(n - 3)f\left(\frac{-x}{2}\right) + \tilde{f}(-2x) = 0
\]

for all \( x \in A \). Since \( \tilde{f}(-x) = \tilde{f}(x) \), we obtain from (2.7) and (2.4) that

\[
2(n - 2)\tilde{f}\left(\frac{x}{2}\right) + (n - 2)\tilde{f}(x) + 2\tilde{f}(2x) = 0
\] (2.8)

for all \( x \in A \). It follows from (2.5) and (2.8) that

\[
2\tilde{f}\left(\frac{x}{2}\right) - \tilde{f}(x) = 0
\] (2.9)

for all \( x \in A \). By using (2.5) and (2.9), we get \( n \tilde{f}(x/2) = 0 \) and so \( f(-x) = -f(x) \) for all \( x \in A \). Hence, we obtain from (2.7) that \( f(x/2) = (1/2)f(x) \) for all \( x \in A \). Letting \( x_1 = x + y, x_2 = -x, x_3 = -y \) and \( x_4 = \cdots = x_n = 0 \) in (2.2), we get

\[
\|f\left(\frac{-y}{2}\right) + f\left(\frac{-x}{2}\right) + (n - 3)f\left(\frac{-x-y}{2}\right) + f\left(\frac{x+y}{2}\right) + (n - 3)f\left(\frac{x}{2}\right) + (n - 3)f\left(\frac{y}{2}\right) + C(n - 3, 2)f(0) + \tilde{f}(x + y)\| \leq \|\lambda f(0)\|
\] (2.10)

for all \( x, y \in A \). Next, notice that, using oddness of \( f \) and \( f(x/2) = (1/2)f(x) \), we have

\[
f(x + y) = f(x) + f(y)
\] (2.11)

for all \( x, y \in A \), as desired.
We can prove the following lemma by the same reasoning as in the proof of Theorem 2.2 of [6].

**Lemma 2.2.** Let \( f : A \to X \) be an additive mapping such that \( f(\mu x) = \mu f(x) \) for all \( \mu \in T^1_{1/\nu_0} \) and all \( x \in A \). Then the mapping \( f \) is \( C \)-linear.

**Theorem 2.3.** Let \( f : A \to X \) be a mapping satisfying \( f(0) = 0 \) and the inequality

\[
\left\| \sum_{1 \leq i < j \leq n} f \left( \frac{\mu x_i + \mu x_j}{2} + \sum_{l=1}^{n-2} \mu x_{k_l} \right) + f \left( \sum_{i=2}^{n} \mu x_i \right) + \mu f(x_1) + \Delta f(a, b) \right\| \leq \left\| \lambda f \left( \sum_{i=1}^{n} \mu x_i \right) \right\| + \delta
\]

(2.12)

for some \( \delta > 0 \), for all \( \mu \in T^1_{1/\nu_0} \) and all \( a,b,x_1,\ldots,x_n \in A \). Then there exists a unique derivation \( \mathcal{D} : A \to X \) such that

\[
\left\| f(x) - \mathcal{D}(x) \right\| \leq \frac{13n - 24}{n(n-4)} \delta
\]

(2.13)

for all \( x \in A \).

**Proof.** Letting \( a = b = 0, x_1 = 2x, x_2 = -2x, x_3 = \cdots = x_n = 0 \) and \( \mu = 1 \) in (2.12), we get

\[
\left\| (n-2) \bar{f}(x) + \bar{f}(2x) \right\| \leq \delta
\]

(2.14)

for all \( x \in X \). Letting \( a = b = 0, x_1 = 2x, x_2 = -x, x_3 = -2x, x_4 = \cdots = x_n = 0 \) and \( \mu = 1 \) in (2.12), we get

\[
\left\| 2f \left( \frac{-x}{2} \right) + (n-3)f(-x) + f(x) + 2(n-3)f \left( \frac{x}{2} \right) + \bar{f}(2x) \right\| \leq \delta
\]

(2.15)

for all \( x \in X \). Letting \( a = b = 0, x_1 = -2x, x_2 = x, x_3 = x, x_4 = \cdots = x_n = 0 \) and \( \mu = 1 \) in (2.12), we get

\[
\left\| 2f \left( \frac{x}{2} \right) + (n-3)f(x) + f(-x) + 2(n-3)f \left( \frac{-x}{2} \right) + \bar{f}(-2x) \right\| \leq \delta
\]

(2.16)

for all \( x \in X \). It follows from (2.15) and (2.16) that

\[
\left\| (n-2) \bar{f} \left( \frac{x}{2} \right) + \frac{(n-2)}{2} \bar{f}(x) + \bar{f}(2x) \right\| \leq \delta
\]

(2.17)

for all \( x \in X \). It follows from (2.14) and (2.17) that

\[
\left\| \bar{f}(x) \right\| \leq \frac{6}{n} \delta
\]

(2.18)
for all $x \in X$. It follows from (2.15) and (2.18) that

$$\left\|2\tilde{f}\left(\frac{x}{2}\right) + f(x) + (n-4)f(-x) + 2(n-4)f\left(\frac{x}{2}\right)\right\| \leq \frac{n+6}{n} \delta$$

(2.19)

for all $x \in X$. From the last two inequalities, we have

$$\left\|f(2x) + 2f(-x)\right\| \leq \frac{n+24}{n(n-4)} \delta$$

(2.20)

for all $x \in X$. It follows from (2.18) and (2.20) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{13n-24}{2n(n-4)} \delta$$

(2.21)

for all $x \in X$. Hence

$$\left\|\frac{1}{2^m}f(2^m x) - \frac{1}{2^m}f(2^m x)\right\| \leq \frac{13n-24}{2n(n-4)} \sum_{k=r}^{m-1} \delta$$

(2.22)

for all $x \in X$ and integers $m > r \geq 0$. Thus it follows that a sequence $\{(1/2^m)f(2^m x)\}$ is Cauchy in $Y$ and so it converges. Therefore we can define a mapping $\mathfrak{D} : X \to Y$ by $\mathfrak{D}(x) := \lim_{m \to \infty}(1/2^m)f(2^m x)$ for all $x \in X$. In addition it is clear from (2.12) that the following inequality:

$$\left\|\sum_{1 \leq i < j \leq n \atop 1 \leq k \neq i, j \leq n} \mathfrak{D} \left(\frac{\mu x_i + \mu x_j}{2} + \sum_{i=1}^{n-2} \mu x_k\right) + \mathfrak{D} \left(\sum_{i=2}^{n} \mu x_i\right) + \mu \mathfrak{D}(x_1)\right\|$$

$$= \lim_{m \to \infty} \frac{1}{2^m} \left\|\sum_{1 \leq i < j \leq n \atop 1 \leq k \neq i, j \leq n} f \left(2^{m-1} \mu(x_i + x_j) + \sum_{i=1}^{n-2} 2^m \mu x_k\right) + f \left(\sum_{i=2}^{n} 2^m \mu x_i\right) + \mu f(2^m x_1)\right\|$$

(2.23)

$$\leq \lim_{m \to \infty} \frac{1}{2^m} \left\|\lambda f \left(\sum_{i=1}^{n} 2^m \mu x_i\right)\right\| + \lim_{m \to \infty} \frac{\delta}{2^m}$$

$$= \left\|\lambda \mathfrak{D} \left(\sum_{i=1}^{n} \mu x_i\right)\right\|$$

holds for all $\mu \in T^{1/2}_{1/n}$ and all $x_1, \ldots, x_n \in X$. If we put $\mu = 1$ in the last inequality, then $\mathfrak{D}$ is additive by Lemma 2.1. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \cdots = x_n = 0$ in last inequality and using Lemma 2.1, we get

$$(n-2)\tilde{\mathfrak{D}}\left(\frac{\mu x}{2}\right) + \mathfrak{D}(-\mu x) + \mu \mathfrak{D}(x) = \mu \mathfrak{D}(x) - \mathfrak{D}(\mu x).$$

(2.24)
So $\mathcal{D}(\mu x) = \mu \mathcal{D}(x)$ for all $x \in X$ and all $\mu \in T_{1/n}^\ast$. Now by using Lemmas 2.1 and 2.2, we infer that the mapping $\mathcal{D} : X \to Y$ is $\mathbb{C}$-linear. Taking the limit as $m \to \infty$ in (2.22) with $r = 0$, we get (2.13).

To prove the afore-mentioned uniqueness, we assume now that there is another $\mathbb{C}$-linear mapping $\mathcal{L} : A \to X$ which satisfies the inequality (2.13). Then we get

$$\left\| \frac{1}{2^m} f(2^m x) - \mathcal{L}(x) \right\| = \frac{1}{2^m} \left\| f(2^m x) - \mathcal{L}(2^m x) \right\| \leq \frac{13n - 24}{2^mn(n-4)} \delta$$

for all $x \in A$ and integers $m \geq 1$. Thus from $m \to \infty$, one establishes

$$\mathcal{D}(x) - \mathcal{L}(x) = 0$$

for all $x \in A$, completing the proof of uniqueness.

Now, we have to show that $\mathcal{D}$ is a derivation. To this end, let $x_1 = x_2 = \cdots = x_n = 0$ in (2.12), we get

$$\left\| f(ab) - f(a)b - af(b) \right\| \leq \delta$$

for all $a, b \in A$. It follows from linearity of $\mathcal{D}$ and (2.27) that

$$\left\| \mathcal{D}(ab) - \mathcal{D}(a)b - a\mathcal{D}(b) \right\| = \left\| \frac{1}{2^m} \mathcal{D}(2^m ab) - \mathcal{D}(a)\frac{1}{2^m}(2^m b) - \frac{1}{2^m}(2^m a)\mathcal{D}(b) \right\|$$

$$= \lim_{m \to \infty} \left\| \frac{1}{4^m} f(4^m ab) - f(4^m a)\frac{1}{4^m}(2^m b) - \frac{1}{4^m}(2^m a) f(2^m b) \right\|$$

$$= \lim_{m \to \infty} \frac{1}{4^m} \left\| f(2^m a 2^m b) - f(2^m a)(2^m b) - (2^m a) f(2^m b) \right\|$$

$$\leq \lim_{m \to \infty} \frac{1}{4^m} \delta$$

$$= 0$$

for all $a, b \in A$. This means that $\mathcal{D}$ is a derivation from $A$ into $X$. Therefore the mapping $\mathcal{D} : A \to X$ is a unique derivation satisfying (2.13), as desired.

\begin{theorem}
Let $A$ be an amenable Banach algebra and let $f : A \to X^\ast$ be a mapping such that $f(0) = 0$ and (2.12). If

$$\sup \{ \| f(x) \| : \| x \| \leq 1 \} < \infty,$$

then there exists $x_0 \in X^\ast$ such that

$$\left\| f(a) - ax_0 - x_0a \right\| \leq \frac{13n - 24}{n(n-4)} \delta$$

for all $a \in A$.
\end{theorem}
Proof. Let sup\{\|f(x)\| : \|x\| \leq 1\} = M_f. Then by (2.29), we have \(M_f < \infty\). By Theorem 2.3, there exists a derivation \(D : A \rightarrow X^*\) satisfying (2.13). Then we have

\[
\sup\{\|D(x)\| : \|x\| \leq 1\} \leq M_f + \frac{13n - 24}{n(n - 4)}\delta.
\]

(2.31)

This means that \(D\) is bounded, and hence \(D\) is continuous. On the other hand, \(A\) is amenable. Then every continuous derivation from \(A\) into \(X^*\) is an inner derivation. It follows that \(D\) is and an inner derivation. In the other words, there exists \(x_0 \in X^*\) such that \(D(a) = ax_0 - x_0a\) for all \(a \in A\). This completes the proof. \(\square\)

We know that every nuclear \(C^*\)-algebra is amenable (see [7]). Then we have the following result.

**Corollary 2.5.** Let \(A\) be a nuclear \(C^*\)-algebra and let \(f : A \rightarrow X^*\) be a mapping such that \(f(0) = 0\), and (2.12) and (2.29). Then there exists \(x_0 \in X^*\) such that

\[
\|f(a) - ax_0 - x_0a\| \leq \frac{13n - 24}{n(n - 4)}\delta
\]

(2.32)

for all \(a \in A\).

**Theorem 2.6.** Let \(A\) be a \(C^*\)-algebra and let \(f : A \rightarrow A^*\) be a mapping such that \(f(0) = 0\), and (2.12) and (2.29). Then there exists \(a' \in A^*\) such that

\[
\|f(a)(b) - a'(ba - ab)\| \leq \frac{13n - 24}{n(n - 4)}\delta\|b\|
\]

(2.33)

for all \(a, b \in A\).

Proof. We know that every \(C^*\)-algebra is weakly amenable (see, e.g., [7]). Then every continuous derivation from \(A\) into \(A^*\) is an inner derivation. By the same reasoning as in the proof of Theorem 2.4, there exists \(a' \in A^*\) such that \(D(a) = aa' - a'a\) for all \(a \in A\), and

\[
\|f(a) - aa' - a'a\| \leq \frac{13n - 24}{n(n - 4)}\delta
\]

(2.34)

for all \(a \in A\). By definition of module actions of \(A\) on \(A^*\), we have

\[
\|f(a)(b) - a'(ba - ab)\| \leq \frac{13n - 24}{n(n - 4)}\delta\|b\|
\]

(2.35)

for all \(a, b \in A\). \(\square\)
Corollary 2.7. Let \( A \) be a commutative \( \mathbb{C}^* \)-algebra and let \( f : A \rightarrow A^* \) be a mapping such that \( f(0) = 0 \), and \((2.12)\) and \((2.29)\). Then

\[
\lim_{m \to \infty} \frac{1}{2m} f(2^m a) = 0, \\
\|f(a)\| \leq \frac{13n - 24}{n(n - 4)} \delta
\]

(2.36)

for all \( a \in A \).

References


