Research Article

Improved Bounds for Restricted Isometry Constants

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The purpose of this paper is to establish improved bounds for restricted isometry constants δk. Our results, to some extent, improve and extend the well-known bound \(\delta_k < 0.307\) in (Cai et al., 2010) to \(\delta_k < 0.308\).

1. Introduction

Consider the following equation:

\[ y = A\beta + z, \]  

(1.1)

where the matrix \(A \in \mathbb{R}^{n \times m} \) (\(n < m\)) and \(z \in \mathbb{R}^n\) is a vector of measurement errors. If \(z = 0\), then (1.1) is an underdetermined linear system with fewer equations than unknowns. The task is to reconstruct the signal \(\beta \in \mathbb{R}^m\) based on the matrix \(A\) and the vector \(y\). Usually, we consider \(\ell_0\) minimization problem:

\[ \min_{\beta \in \mathbb{R}^m} \|\hat{\beta}\|_0, \quad \text{subject to } y = A\hat{\beta} + z \text{ and } \|z\|_2 \leq \epsilon, \]  

(1.2)

where \(\|\cdot\|_0\) denotes the \(\ell_0\)-norm of a vector, that is, the number of its nonzero components.
We need to solve this problem and find the sparsest solution among all the possible solutions. But it requires a combinatorial search and remains an NP-hard problem that cannot be solved in practice. Naturally, an alternative strategy is to find \( \ell \)-solutions. But it requires a combinatorial search and remains an NP-hard problem that cannot be solved in practice.

\[
\min_{\beta \in \mathbb{C}} \| \hat{\beta} \|_1, \quad \text{subject to } y = A\hat{\beta} + z \text{ and } \| z \|_2 \leq \epsilon,
\]

and we expect to find the sparsest solution.

In order to exactly recover the sparsest solution in \( \ell_1 \) minimization, Candès and Tao [1] introduced restricted isometry property (see Restricted Isometry Constants in Definition 2.1). So far, there are various methods [1–9] to give the sufficient conditions on \( \delta_k \): Candès [3] established that \( \delta_{2k} < \sqrt{2} - 1 \approx 0.4142 \) is the sufficient condition of exactly recover \( k \)-sparse vectors via \( \ell_1 \) minimization (a vector \( x \) is \( k \)-sparse if \( \| x \|_0 \leq k \)). This sufficient condition was later improved to \( \delta_{2k} < 2(3 - \sqrt{2})/7 \approx 0.4531 \) in [6] and to \( \delta_{2k} < 3/(4 + \sqrt{6}) \approx 0.4652 \) in [5]. Later, the sufficient condition was improved to \( \delta_{2k} < 1/(1 + \sqrt{1.25}) \approx 0.4721 \) in [10] for the special case that \( k \) is a multiple of 4 or \( k \) is very large and to \( \delta_{2k} < 4/(6 + \sqrt{6}) \approx 0.4734 \) in [5]. Naturally, we want to give the sufficient condition about \( \delta_k \). To the best of our knowledge, T. T. Cai et al. [2] firstly show that the restricted isometry constant \( \delta_k \) of \( A \) satisfies \( \delta_k < 0.307 \) for general \( k \), then \( k \)-sparse signals are guaranteed to be recovered exactly via \( \ell_1 \) minimization. Based on this motivation, we construct a different partition of \( \{1, 2, \ldots, m\} \) and then discuss the error between original signal \( \beta \) and recover signal \( \hat{\beta} \) in (1.3). The main work of this paper is to improve the condition to \( \delta_k < 0.308 \) and to prove that the \( k \)-sparse signals can be recovered exactly via \( \ell_1 \) minimization in no noise case and be estimated stably under the perturbation of noise.

To state our main results, we firstly give the following preliminaries.

2. Preliminaries

In 2005, Candès and Tao [1] firstly present the definition of the restricted isometry constant.

**Definition 2.1** (see [1], restricted isometry constants). Let \( F \) be the matrix with finite collection of vectors \( (v_j)_{j \in J} \in \mathbb{R}^n \) as columns. For every integer \( 1 \leq S \leq |J| \), the \( S \)-restricted isometry constants \( \delta_S \) are defined as the smallest quantity such that \( F_T \) obeys

\[
(1 - \delta_S)\|c\|_2^2 \leq \|F_Tc\|_2^2 \leq (1 + \delta_S)\|c\|_2^2
\]

for all subsets \( T \subset J \) of cardinality at most \( S \) and all real coefficients \( (c_j)_{j \in T} \). Similarly, we define the \( S,S' \)-restricted orthogonality constants \( \theta_{S,S'} \) for \( S + S' \leq |J| \) to be the smallest quantity such that \( \langle F_Tc, F_{T'}c' \rangle \leq \theta_{S,S'}\|c\|_2 \cdot \|c'\|_2 \) holds for all disjoint sets \( T, T' \subset J \) of cardinality \( |T| \leq S \) and \( |T'| \leq S' \).
In addition, we can easily check the following monotone properties:

\[ \delta_k \leq \delta_{k_1}, \quad \text{if} \quad k \leq k_1 \leq n, \]
\[ \theta_{k,k'} \leq \theta_{k_1,k_1'}, \quad \text{if} \quad k \leq k_1, k' \leq k_1', k_1 + k_1' \leq n. \] (2.3)

Apart from the above relationship, Candès and Tao [1] proved that the restricted orthogonality constant \( \theta_{k,k'} \) and the restricted isometry constant \( \delta_k \) are related by the following lemma.

**Lemma 2.2 (see [1]).** One has \( \theta_{S,S'} \leq \delta_{S+S'} \leq \theta_{S,S} + \max(\delta_S, \delta_{S'}) \) for all \( S, S' \).

In the sequel, a useful inequality between \( \ell_1 \)-norm and \( \ell_2 \)-norm will be introduced.

**Proposition 2.3 (see [2]).** For any \( x \in \mathbb{R}^n \),

\[ \|x\|_2 - \frac{\|x\|_1}{\sqrt{n}} \leq \sqrt{n} \left( \frac{\max|t| - \min|t|}{4} \right). \] (2.4)

At the last of preliminaries, we introduce the square root lifting inequality [10].

**Lemma 2.4 (see [10]).** For any \( a \geq 1 \) and positive integers \( k, k' \) such that \( ak' \) is an integer, then

\[ \theta_{k,ak'} \leq \sqrt{a} \theta_{k,k'}. \] (2.5)

### 3. Improved Bounds for Restricted Isometry Constants

In this section, we discuss the new restricted isometry constant \( \delta_k \) for sparse signal recovery via \( \ell_1 \) minimization in (1.3).

**Theorem 3.1.** Suppose \( \beta \) is \( k \)-sparse. Then the \( \ell_1 \) minimizer \( \hat{\beta} \) defined in (1.3) satisfies

\[ \|\beta - \hat{\beta}\|_2 \leq \frac{2\sqrt{2}\sqrt{1 + \delta_k}}{1 - 13/4\delta_k} \epsilon, \] (3.1)

where \( \delta_k \) is the \( k \)-restricted isometry constant of \( A \) in (1.3).

**Proof.** Let \( h = \beta - \hat{\beta} \in \mathbb{R}^m \). Partition \( \{1, 2, \ldots, m\} \) into the following sets:

\[ T_0 = \{1, 2, \ldots, k\}, \quad T_1 = \{k + 1, \ldots, k + \frac{k}{2}\}, \quad T_2 = \{k + \frac{k}{2} + 1, \ldots, 2k\}, \ldots \] (3.2)
where \( k \) is an even number. And rearranging the indices if necessary, \(|h(1)| \geq |h(2)| \geq \cdots\), where \(|h(i)|, i = 1, 2, \ldots, m\) is the \( i \)th entry of the above vector by rearranging the indices. Then by Proposition 2.3, we obtain

\[
\sum_{i \geq 1} \|h_T^i\|_2 \leq \frac{1}{\sqrt{k/2}} \sum_{i \geq 1} \|h_T^i\|_1 + \frac{\sqrt{k/2}}{4} \left( |h(k + 1)| - \left| h\left( \frac{k}{2} \right) \right| \right) + \cdots. \tag{3.3}
\]

By the triangle inequality for \( \|\cdot\|_1 \), we have

\[
\|\beta\|_1 - \|h_T^0\|_1 \leq \|\hat{\beta} + h_T^0\|_1. \tag{3.4}
\]

Since \( T_0 \cap T_0^c = \emptyset \), we have

\[
\|\beta\|_1 - \|h_T^0\|_1 + \|h_T^0\|_1 = \|\hat{\beta} + h_T^0 + h_T^0\|_1 = \|\hat{\beta} + h\|_1 = \|\hat{\beta}\|_1 \leq \|\beta\|_1. \tag{3.5}
\]

The last inequality holds because \( \hat{\beta} \) solves (1.3). Then the result is that

\[
\|h_T^0\|_1 \leq \|h_T^0\|_1. \tag{3.6}
\]

Substituting (3.6) into (3.3), we get

\[
\sum_{i \geq 1} \|h_T^i\|_2 \leq \frac{1}{\sqrt{k/2}} \left( \|h_T^0\|_1 + \frac{\sqrt{k/2}}{4} |h(k + 1)| \right)
\leq \frac{1}{\sqrt{k/2}} \|h_T^0\|_1 + \frac{\sqrt{k/2}}{4} \|h_T^0\|_2 \sqrt{k}
\leq \frac{1}{\sqrt{k/2}} \cdot \sqrt{k} \|h_T^0\|_2 + \frac{1}{4} \|h_T^0\|_2
\leq \frac{9\sqrt{2}}{8} \|h_T^0\|_2. \tag{3.7}
\]

And note that

\[
|\langle A\beta, A\hat{\beta} \rangle| \geq |\langle A\hat{\beta}, A\hat{\beta} \rangle| - \sum_{i \geq 1} |\langle A\beta, A\hat{\beta} \rangle|. \tag{3.8}
\]

From (2.2) and (2.5) in Lemma 2.4, we have

\[
|\langle A\beta, A\hat{\beta} \rangle| \leq \theta_{k/2,k} \|h_T^0\|_2 \cdot \|h_T^0\|_2. \tag{3.9}
\]

By Lemma 2.2, we have

\[
\theta_{k/2,k} = \theta_{k/2,k/2} \leq \sqrt{2}\delta_{k/2,k/2} = \sqrt{2}\delta_k. \tag{3.10}
\]
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From (3.7)–(3.10), we have

\[
\langle Ah, Ah^T_0 \rangle \geq \left(1 - \delta_k\right)\|h_k\|_2^2 \theta_{k/2,k}^{i=1} \|h_i\|_2
\]

\[
\geq \left(1 - \delta_k\right)\|h_k\|_2^2 - \sqrt{2}\delta_k\|h_k\|_2 \cdot 9\sqrt{2}/8\|h_k\|_2 \geq \left(1 - \frac{13\delta_k}{4}\right)\|h_k\|_2.
\]

From (1.3), we have

\[
\|Ah\|_2 = \left\|A(\hat{\beta} - \beta)\right\|_2 \leq \left\|A\hat{\beta} - y\right\|_2 + \left\|A\beta - y\right\|_2 \leq 2\epsilon.
\]

In addition, we obtain the following relation by simple calculation

\[
\left\|h_{T_0}\right\|_2^2 = \left(h(k+1)^2 + h(k+2)^2 + \cdots\right)
\]

\[
\leq \max_{i \geq k+1} |h(i)| \cdot (|h(k+1)| + |h(k+2)| + \cdots)
\]

\[
= \max_{i \geq k+1} |h(i)| \cdot \left\|h_{T_0}\right\|_1
\]

\[
\leq \frac{\|h_{T_0}\|_1}{k} \cdot \left\|h_{T_0}\right\|_1.
\]

Since \(\|h_{T_0}\|_1 \leq \|h_{T_0}\|_2\), we have

\[
\left\|h_{T_0}\right\|_2 \leq \frac{\|h_{T_0}\|_2^2}{k}.
\]

By the norm inequality \(\|h_{T_0}\|_2^2 \leq k\|h_{T_0}\|_2^2\) and (3.14), we have

\[
\left\|h_{T_0}\right\|_2 \leq \|h_{T_0}\|_2.
\]

From (3.7), (3.11)–(3.12), and (3.15), we have

\[
\|h\|_2 \leq \sqrt{2}\|h_{T_0}\|_2 \leq \frac{\sqrt{2}|\langle Ah, Ah^T_0 \rangle|}{(1 - 13\delta_k/4)\|h_{T_0}\|_2} \leq \frac{\sqrt{2}\|Ah\|_2 \cdot \|Ah^T_0\|_2}{(1 - 13\delta_k/4)\|h_{T_0}\|_2}
\]

\[
\leq \frac{\sqrt{2} \cdot 2\epsilon \cdot \sqrt{1 + \delta_k}\|h_{T_0}\|_2}{(1 - 13\delta_k/4)\|h_{T_0}\|_2} \leq 2\sqrt{2}\frac{\sqrt{1 + \delta_k}}{1 - 13\delta_k/4} \epsilon.
\]

Remark 3.2. If \(\epsilon = 0\), it is the case where the \(k\)-sparse signals are guaranteed to be recovered exactly via \(\ell_1\) minimization under no noise situation.
Corollary 3.3. Let \( y = A\beta + z \) with \( \|z\|_2 \leq \epsilon \). Suppose \( \beta \) is \( k \)-sparse with \( k > 1 \). Then under the condition \( \delta_k < 0.308 \) the constrained \( \ell_1 \) minimizer \( \hat{\beta} \) given in (1.3) satisfies

\[
\|\beta - \hat{\beta}\|_2 \leq \frac{3.2344}{0.308 - \delta_k} \epsilon.
\] (3.17)

Proof. The proof of this corollary can be easily obtained if we put \( \delta_k < 0.308 \) into the inequality (3.1) in Theorem 3.1.

\[ \square \]

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References

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