Research Article

On Reciprocal Series of Generalized Fibonacci Numbers with Subscripts in Arithmetic Progression

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We investigate formulas for closely related series of the forms:

\[ \sum_{n=0}^{\infty} \frac{1}{U_{an+b} + c}, \]

\[ \sum_{n=0}^{\infty} (-1)^n U_{an+b} / (U_{an+b} + c)^2, \]

\[ \sum_{n=0}^{\infty} U_{2(an+b)} / (U_{2(an+b)} + c)^2 \]

for certain values of \( a, b, \) and \( c. \)

1. Introduction

Let \( p \) be a nonzero integer such that \( \Delta = p^2 + 4 \neq 0. \) The generalized Fibonacci and Lucas sequences are defined by the following recurrences:

\[ U_{n+1} = pU_n + U_{n-1}, \]
\[ V_{n+1} = pV_n + V_{n-1}, \]

where \( U_0 = 0, U_1 = 1 \) and \( V_0 = 2, V_1 = p, \) respectively. When \( p = 1, U_n = F_n \) (\( n \)th Fibonacci number) and \( V_n = L_n \) (\( n \)th Lucas number).

If \( \alpha \) and \( \beta \) are the roots of equation \( x^2 - px - 1 = 0, \) the Binet formulas of the sequences \( \{U_n\} \) and \( \{V_n\} \) have the forms:

\[ U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n, \]

respectively.

In [1], Backstrom developed formulas for closely related series of the form:

\[ \sum_{n=0}^{\infty} \frac{1}{F_{an+b} + c}. \]
for certain values of $a$, $b$, and $c$. For example, he obtained the following series:

$$
\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_{K}} = \frac{K \sqrt{5}}{2L_{K}},
$$

$$
\sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)K+2t} + F_{K}} = \begin{cases} 
\frac{(\sqrt{5} - 5F_{t}/L_{t})}{2L_{K}}, & t \text{ even}, \\
\frac{(\sqrt{5} - L_{t}/F_{t})}{2L_{K}}, & t \text{ odd},
\end{cases}
$$

where $K$ represents an odd integer and $t$ is an integer in the range $-(K - 1)/2$ to $(K - 1)/2$ inclusive. Also, he gave the similar results for Lucas numbers.

In [2], Popov found in explicit form series of the form:

$$
\sum_{n=0}^{\infty} \frac{1}{F_{an+b} \pm c}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{an+b}F_{cn+d}}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{an+b} \pm F_{cn+d}}',
$$

for certain values of $a$, $b$, $c$, and $d$.

In [3], Popov generalized some formulas of Backstrom [1] related to sums of reciprocal series of Fibonacci and Lucas numbers. For example,

$$
\Delta \sum_{n=0}^{\infty} \frac{(-q)^{s}nr}{V_{(2n+1)r+2s} - (-q)^{r}nr} V_{r} = \begin{cases} 
\frac{\beta^{s}}{U_{s}U_{r}}, & |\beta| \left|\frac{a}{\beta}\right| < 1, \\
\frac{\alpha^{s}}{U_{s}U_{r}}, & |\alpha| \left|\frac{a}{\alpha}\right| < 1,
\end{cases}
$$

where $s$ and $r$ are integers.

In [4], Gauthier found the closed form expressions for the following sums:

$$
\sum_{k=0}^{m} \frac{(-1)^{kn} f_{(2k+1)n}}{f_{(k+1)n} f_{kn}}, \quad m, n \geq 1,
$$

$$
\sum_{k=0}^{m} \frac{(-1)^{kn} f_{(2k+1)n}}{f_{(k+1)n}^{2} f_{kn}^{2}}, \quad m, n \geq 0,
$$

where for $x \neq 0$ an indeterminate, the generalized Fibonacci and Lucas polynomials $\{f_{n}\}_{n}$ and $\{l_{n}\}_{n}$ are given by the following recurrences:

$$
\begin{align*}
f_{n+2} &= xf_{n+1} + f_{n}, & f_{0} = 0, & f_{1} = 1, & n \geq 0, \\
l_{n+2} &= xl_{n+1} + l_{n}, & l_{0} = 2, & l_{1} = x, & n \geq 0,
\end{align*}
$$

respectively.

In this paper, we investigate formulas for closely related series of the forms:

$$
\sum_{n=0}^{\infty} \frac{1}{U_{an+b} + c'}, \quad \sum_{n=0}^{\infty} \frac{(-1)^{n} U_{an+b}}{(U_{an+b} + c)^{2}}, \quad \sum_{n=0}^{\infty} \frac{U_{2(an+b)}}{(U_{an+b} + c)^{2}},
$$

for certain values of $a$, $b$ and $c$. 
2. On Some Series of Reciprocals of Generalized Fibonacci Numbers

In this section, firstly, we will give the following lemmas for further use.

**Lemma 2.1.** Let \( n \) be an arbitrary nonzero integer. For integer \( m \geq 1 \),

\[
\sum_{k=1}^{m} \frac{(-1)^{kn}U_{(2k+1)n}}{U_{(k+1)n}U_{kn}} = \frac{1}{4U_n} \left( \frac{V_{n}^2}{U_{n}} - \frac{V_{(m+1)n}^2}{U_{(m+1)n}} \right),
\]

and for integer \( m \geq 0 \),

\[
\sum_{k=0}^{m} \frac{(-1)^{kn}U_{(2k+1)n}}{V_{(k+1)n}V_{kn}} = \frac{U_{(m+1)n}}{4U_n V_{(m+1)n}^2}.
\]

**Proof.** We give the proof of Lemma 2.1 as the proofs of the sums in [4], using the following equalities:

\[
\frac{U_{(2k+1)n}}{U_{(k+1)n}U_{kn}} = \frac{1}{2} \left( \frac{V_{kn}}{U_{kn}} + \frac{V_{(k+1)n}}{U_{(k+1)n}} \right),
\]

\[
\frac{U_{(2k+1)n}}{V_{(k+1)n}V_{kn}} = \frac{1}{2} \left( \frac{U_{(k+1)n}}{V_{(k+1)n}} + \frac{U_{kn}}{V_{kn}} \right),
\]

\[
\frac{(-1)^{kn}U_{n}}{U_{(k+1)n}U_{kn}} = \frac{1}{2} \left( \frac{V_{kn}}{U_{kn}} - \frac{V_{(k+1)n}}{U_{(k+1)n}} \right),
\]

\[
\frac{(-1)^{kn}U_{n}}{V_{(k+1)n}V_{kn}} = \frac{1}{2} \left( \frac{U_{(k+1)n}}{V_{(k+1)n}} - \frac{U_{kn}}{V_{kn}} \right).
\]

**Lemma 2.2.** For arbitrary integers \( n \) and \( t \),

\[
V_{2n} - (-1)^{n-t}V_{2t} = \Delta U_{n-t}U_{n+t},
\]

\[
V_{2n} + (-1)^{n-t}V_{2t} = V_{n-t}V_{n+t},
\]

\[
U_{n}^2 - (-1)^{n-t}U_{t}^2 = \Delta U_{n-t}U_{n+t},
\]

\[
V_{n}^2 - (-1)^{n-t}V_{t}^2 = \Delta U_{n-t}U_{n+t}.
\]

**Proof.** From Binet formulas of sequences \( \{U_n\} \) and \( \{V_n\} \), the desired results are obtained.

**Theorem 2.3.** For an odd integer \( t \),

\[
\sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} + U_{t}} = \frac{1}{2V_{t}} \left( \frac{2 - V_{2t}}{U_{2t}} - \frac{2 - V_{2(m+1)t}}{U_{2(m+1)t}} \right),
\]

\[
\sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} - U_{t}} = \frac{1}{2V_{t}} \left( \frac{2 + V_{2t}}{U_{2t}} - \frac{2 + V_{2(m+1)t}}{U_{2(m+1)t}} \right).
\]
Proof. By replacing $n$ with $(2n + 1)t$ in (2.5), we have

$$U_{(2n+1)t}^2 - U_t^2 = U_{2nt}U_{2(n+1)t}, \quad (2.7)$$

or

$$\frac{1}{U_{(2n+1)t} + U_t} = \frac{1}{U_{2nt} + U^2_{(n+1)t}} \cdot \frac{U_t - (-1)^t}{U_{2(n+1)t}U_{2(n+1)t}}. \quad (2.8)$$

Taking $r = (2n + 1)t$ and $s = t$ in the equality $V_sU_r = U_{r+s} + (-1)^sU_{r-s}$ [5], the equality (2.8) is rewritten as follows:

$$\frac{1}{U_{(2n+1)t} + U_t} = \frac{1}{U_{2nt} + U^2_{(n+1)t}} \cdot \frac{1}{U_{2(n+1)t}}, \quad (2.9)$$

We have the sum

$$\sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} + U_t} = \frac{1}{V_t} \sum_{n=1}^{m} \left( \frac{1}{U_{2nt} + U^2_{(n+1)t}} \cdot \frac{1}{U_{2(n+1)t}} \right) \cdot \frac{1}{U_{2(n+1)t}}. \quad (2.10)$$

For an odd integer $t$, we have

$$\sum_{n=1}^{m} \left( \frac{1}{U_{2nt} - U^2_{(n+1)t}} \right) = \frac{1}{U_{2t}} - \frac{1}{U_{2(n+1)t}}. \quad (2.11)$$

and taking $s = 2nt$ and $r = 2t$ in identity [5]:

$$U_{s+r}V_s - U_sV_{s+r} = 2(-1)^sU_r, \quad (2.12)$$

we get

$$\sum_{n=1}^{m} \frac{1}{U_{2nt}U_{2(n+1)t}} = \frac{1}{2U_{2t}} \sum_{n=1}^{m} \left( \frac{V_{2nt} - V_{2(n+1)t}}{U_{2nt} - U^2_{(n+1)t}} \right) = \frac{1}{2U_{2t}} \left( \frac{V_{2t}}{U_{2t} - U^2_{2(m+1)t}} \right). \quad (2.13)$$

Substituting (2.11) and (2.13) in (2.10), we have the desired result. \(\square\)

For example, if we take $t = 1$ and $p = 1$ in (2.6), we have

$$\sum_{n=1}^{m} \frac{1}{F_{2n+1} + 1} = \frac{F_{2m+1} - 1}{F_{2(m+1)}}. \quad (2.14)$$

Note that

$$F_{2(m+1)} \sum_{n=1}^{m} \frac{1}{F_{2n+1} + 1} = \sum_{n=1}^{m} F_{2n}. \quad (2.15)$$
Corollary 2.4. For an odd integer $t$,

$$
\sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} + U_t} = \begin{cases} 
\frac{1}{2V_t} \left( \frac{V_{(m+1)t} - V_t}{U_{(m+1)t} U_t} \right), & m \text{ is even,} \\
\frac{1}{2V_t} \left( \frac{\Delta U_{(m+1)t} - V_t}{V_{(m+1)t} U_t} \right), & m \text{ is odd,}
\end{cases}
$$

and

$$
\sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} - U_t} = \begin{cases} 
\frac{\Delta}{2V_t} \left( \frac{V_{(m+1)t} U_t - U_{(m+1)t}}{V_t} \right), & m \text{ is even,} \\
\frac{1}{2V_t} \left( \frac{\Delta U_t - V_{(m+1)t}}{V_t U_{(m+1)t}} \right), & m \text{ is odd.}
\end{cases}
$$

Proof. Using the equalities $V_{2n} = V_n^2 - 2(-1)^n = \Delta U_n^2 + 2(-1)^n$ and $U_{2n} = U_n V_n$ in Theorem 2.3, the results are obtained. □

Corollary 2.5. Let $t$ be an odd integer. For $|\beta/\alpha| < 1, t > 0$ and $|\alpha/\beta| < 1, t < 0$,

$$
\sum_{n=1}^{\infty} \frac{1}{U_{(2n+1)t} + U_t} = \frac{1}{2V_t} \left( \sqrt{\Delta} - \frac{V_t}{U_t} \right),
$$

and

$$
\sum_{n=1}^{\infty} \frac{1}{U_{(2n+1)t} - U_t} = \frac{\Delta}{2V_t} \left( \frac{V_t U_t - U_{(m+1)t}}{V_t} \right),
$$

and for $|\beta/\alpha| < 1, t < 0$ and $|\alpha/\beta| < 1, t > 0$,

$$
\sum_{n=1}^{\infty} \frac{1}{U_{(2n+1)t} + U_t} = \frac{-1}{2V_t} \left( \sqrt{\Delta} + \frac{V_t}{U_t} \right),
$$

and

$$
\sum_{n=1}^{\infty} \frac{1}{U_{(2n+1)t} - U_t} = \frac{\Delta}{2V_t} \left( \frac{V_t U_t + U_{(m+1)t}}{V_t} \right).
$$

Proof. Since

$$
\lim_{n \to \infty} \left( \frac{V_{an+b}}{U_{an+b}} \right) = \begin{cases} 
\sqrt{\Delta}, & |\beta/\alpha| < 1, a > 0, \left| \frac{\alpha}{\beta} \right| < 1, a < 0, \\
-\sqrt{\Delta}, & |\beta/\alpha| < 1, a < 0, \left| \frac{\alpha}{\beta} \right| < 1, a > 0,
\end{cases}
$$

the results are easily seen by equalities (2.16). □

Theorem 2.6. For an integer $m \geq 1$ and an arbitrary nonzero integer $t$,

$$
\sum_{n=1}^{m} \frac{(-1)^{nt} U_{(2n+1)t}}{(V_{(2n+1)t} - (-1)^n V_t)^2} = \frac{1}{4\Delta^2 U_t} \left( \frac{V_t^2}{U_t} - \frac{V_{(m+1)t}^2}{U_{(m+1)t}} \right).
$$
Proof. By replacing $n$ with $(2n + 1)t/2$ and $t$ with $t/2$ in (2.4), we have
\[ V_{(2n+1)t} - (-1)^n V_t = \Delta U_{nt} U_{(n+1)t}, \]  
(2.21) or
\[ \frac{1}{V_{(2n+1)t} - (-1)^n V_t} = \frac{1}{\Delta U_{nt} U_{(n+1)t}}. \]  
(2.22)

Multiplying equality (2.22) by \((-1)^n U_{(2n+1)t}/U_{nt} U_{(n+1)t}\), we get
\[ \frac{(-1)^n U_{(2n+1)t}}{U_{nt} U_{(n+1)t}(V_{(2n+1)t} - (-1)^n V_t)} = \frac{(-1)^n U_{(2n+1)t}}{\Delta U_{nt}^2 U_{(n+1)t}^2}. \]  
(2.23)

We have the sum:
\[ \sum_{n=1}^{m} \frac{(-1)^n U_{(2n+1)t}}{U_{nt} U_{(n+1)t}(V_{(2n+1)t} - (-1)^n V_t)} = \frac{1}{\Delta} \sum_{n=1}^{m} \frac{(-1)^n U_{(2n+1)t}}{U_{nt}^2 U_{(n+1)t}^2}. \]  
(2.24)

Using the equalities (2.1) and (2.21), the proof is obtained. \(\square\)

**Corollary 2.7.** For an arbitrary nonzero integer $t$,
\[ \sum_{n=1}^{\infty} \frac{(-1)^n U_{(2n+1)t}}{(V_{(2n+1)t} - (-1)^n V_t)^2} = \frac{1}{4\Delta^2 U_t} \left( \frac{V_t^2}{U_t^2} - \Delta \right). \]  
(2.25)

**Proof.** Taking $m \rightarrow \infty$ in Theorem 2.6 and using (2.19), the result is easily obtained. \(\square\)

**Theorem 2.8.** For an integer $m \geq 0$ and an arbitrary nonzero integer $t$,
\[ \sum_{n=0}^{m} \frac{(-1)^n U_{(2n+1)t}}{(V_{(2n+1)t} + (-1)^n V_t)^2} = \frac{U_{(m+1)t}^2}{4U_t V_{(m+1)t}^2}. \]  
(2.26)

**Proof.** The proof of the theorem is similar to the proof of Theorem 2.6. \(\square\)

**Corollary 2.9.** For an arbitrary nonzero integer $t$,
\[ \sum_{n=0}^{\infty} \frac{(-1)^n U_{(2n+1)t}}{(V_{(2n+1)t} + (-1)^n V_t)^2} = \frac{1}{4\Delta U_t}. \]  
(2.27)

**Proof.** Taking $m \rightarrow \infty$ in Theorem 2.8 and using (2.19), the result is easily obtained. \(\square\)

For example, if we take $t = 3$ and $p = 1$ in (2.27), we have
\[ \sum_{n=0}^{\infty} \frac{(-1)^n F_{3(2n+1)}}{(L_{3(2n+1)} + (-1)^n 4)^2} = \frac{1}{40}. \]  
(2.28)
Theorem 2.10. For an integer \( m \geq 1 \) and an arbitrary nonzero integer \( t \),
\[
\sum_{n=1}^{m} \frac{U_{2(2n+1)t}}{\left(U_{(2n+1)t}^2 - U_t^2\right)^2} = \frac{1}{4U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \frac{V_{2(m+1)t}^2}{U_{2(m+1)t}^2}\right),
\]
\[
\sum_{n=1}^{m} \frac{U_{2(2n+1)t}}{\left(V_{(2n+1)t}^2 - V_t^2\right)^2} = \frac{1}{4\Delta^2U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \frac{V_{2(m+1)t}^2}{U_{2(m+1)t}^2}\right).
\]

Proof. The proof of theorem is similar to the proof of Theorem 2.6. \( \square \)

Corollary 2.11. For an arbitrary nonzero integer \( t \),
\[
\sum_{n=1}^{\infty} \frac{U_{2(2n+1)t}}{\left(U_{(2n+1)t}^2 - U_t^2\right)^2} = \frac{1}{4U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \Delta\right),
\]
\[
\sum_{n=1}^{\infty} \frac{U_{2(2n+1)t}}{\left(V_{(2n+1)t}^2 - V_t^2\right)^2} = \frac{1}{4\Delta^2U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \Delta\right).
\]

Proof. Taking \( m \to \infty \) in Theorem 2.10 and using (2.19), the result is easily obtained. \( \square \)

For example, if we take \( t = 2 \) in the equality (2.30), we have
\[
\sum_{n=1}^{\infty} \frac{U_{4(2n+1)}}{\left(V_{2(2n+1)}^2 - V_2^2\right)^2} = \frac{1}{\Delta^2U_4^3}.
\]

References
