Research Article

Some Identities on Bernoulli and Hermite Polynomials Associated with Jacobi Polynomials

Taekyun Kim,¹ Dae San Kim,² and Dmitry V. Dolgy³

¹ Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
² Department of Mathematics, Sogang University, Seoul, Republic of Korea
³ Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea

Correspondence should be addressed to Dae San Kim, dskim@sogong.ac.kr

Received 17 July 2012; Accepted 9 August 2012

Academic Editor: Josef Diblik

Copyright © 2012 Taekyun Kim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate some identities on the Bernoulli and the Hermite polynomials arising from the orthogonality of Jacobi polynomials in the inner product space $P_n$.

1. Introduction

For $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\beta > -1$, the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ are defined as

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(1 + \alpha + \beta + n)_k}{(\alpha + 1)_k} \left(\frac{x - 1}{2}\right)^k,$$

(see [1–4]), where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \Gamma(\alpha + n)/\Gamma(\alpha)$.

From (1.1), we note that

$$P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(\alpha + 1 + n)}{n!\Gamma(\alpha + \beta + n + 1)} \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha + \beta + n + k + 1)}{\Gamma(\alpha + k + 1)} \left(\frac{x - 1}{2}\right)^k.$$

(1.2)
By (1.2), we see that \( P_n^{(\alpha,\beta)}(x) \) is polynomial of degree \( n \) with real coefficients. It is not difficult to show that the leading coefficient of \( P_n^{(\alpha,\beta)}(x) \) is \( 2^{-n} (\binom{\alpha+\beta+2n}{n}) \). From (1.2), we have

\[
P_n^{(\alpha,\beta)}(1) = \binom{\alpha+\beta}{n}.
\]

By (1.1), we get

\[
\left( \frac{d}{dx} \right)^k P_n^{(\alpha,\beta)}(x) = 2^{-k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} P_{n-k}^{(\alpha+k,\beta+k)}(x)
\]

\[
= \frac{1}{2^k} (n + \alpha + \beta + k)(n + \alpha + \beta + k - 1) \cdots (n + \alpha + \beta + 1) P_{n-k}^{(\alpha+k,\beta+k)}(x),
\]

where \( k \) is a positive integer (see [1–4]).

The Rodrigues’ formula for \( P_n^{(\alpha,\beta)}(x) \) is given by

\[
(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \left( \frac{d}{dx} \right)^k (1-x)^{n-\alpha}(1+x)^{n+\beta}.
\]

It is easy to show that \( u = P_n^{(\alpha,\beta)}(x) \) is a solution of the following differential equation:

\[
\left( 1-x^2 \right) u'' + \left( \beta - \alpha - (\alpha + \beta + 2)x \right) u' + n(n + \alpha + \beta + 1) u = 0.
\]

As is well known, the generating function of \( P_n^{(\alpha,\beta)}(x) \) is given by

\[
F(x,t) = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n = \frac{2^{\alpha+\beta}}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}},
\]

where \( R = \sqrt{1-2xt+t^2} \), (see [1–4]).

From (1.3), (1.4), and (1.6), we can derive the following identity:

\[
\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)} \delta_{n,m},
\]

where \( \delta_{n,m} \) is the Kronecker symbol.

Let \( P_n = \{ p(x) \in \mathbb{R}[x] \mid \deg p(x) \leq n \} \). Then \( P_n \) is an inner product space with respect to the inner product \( \langle q_1(x), q_2(x) \rangle = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} q_1(x)q_2(x) dx \), where \( q_1(x), q_2(x) \in P_n \). From (1.7), we note that \( \{ P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_n^{(\alpha,\beta)}(x) \} \) is an orthogonal basis for \( P_n \).

The so-called Euler polynomials \( E_n(x) \) may be defined by means of

\[
\frac{2}{e^t+1} e^{xt} = e^{E_n(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]
(see [5–22]), with the usual convention about replacing \( E^n(x) \) by \( E_n(x) \). In the special case, \( x = 0 \), \( E_n(0) = E_n \) are called the Euler numbers.

The Bernoulli polynomials are also defined by the generating function to be

\[
\frac{t}{e^t - 1} e^{tx} = e^{B_n(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\]

(1.9)

(see [11–21]), with the usual convention about replacing \( B^n(x) \) by \( B_n(x) \).

From (1.8) and (1.9), we note that

\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k} x^k. \tag{1.10}
\]

For \( n \in \mathbb{Z}_+ \), we have

\[
\frac{dE_n(x)}{dx} = nB_{n-1}(x), \quad \frac{dB_n(x)}{dx} = nE_{n-1}(x) \tag{1.11}
\]

(see [23–29]) By the definition of Bernoulli and Euler polynomials, we get

\[
B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad E_0 = 1, \quad E_n(1) + E_n = 2\delta_{0,n}. \tag{1.12}
\]

In this paper we give some interesting identities on the Bernoulli and the Hermite polynomials arising from the orthogonality of Jacobi polynomials in the inner product space \( P_n \).

### 2. Bernoulli, Euler and Jacobi Polynomials

From (1.4), we have

\[
P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \beta}{k} \left( \frac{x - 1}{2} \right)^{n-k} \left( \frac{x + 1}{2} \right)^{-k}. \tag{2.1}
\]

By (2.1), we have

\[
\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = \frac{1}{2\pi i} \oint \frac{(1 + ((x + 1)/2)z)^{n+\alpha}(1 + ((x - 1)/2)z)^{n+\beta}}{z^{n+1}} dz, \tag{2.2}
\]

where we assume \( x \neq \pm 1 \) and circle around 0 is taken so small that \(-2(x \pm 1)^{-1}\) lie neither on it nor in its interior. It is not so difficult to show that \( P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \).

For \( q(x) \in P_n \), let

\[
q(x) = \sum_{k=0}^{n} C_k P_k^{(\alpha,\beta)}(x), \quad (C_k \in \mathbb{R}). \tag{2.3}
\]
Thus, by (2.4), we get

\[
C_k = \frac{(2k + \alpha + \beta + 1)\Gamma(\alpha + \beta + k + 1)k!}{2^{\alpha + \beta + 1}\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)} \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta P_k^{(\alpha,\beta)}(x)q(x)dx. \tag{2.5}
\]

Thus, by (2.4), we get

\[
\left< q(x), P_k^{(\alpha,\beta)}(x) \right> = C_k \left< P_k^{(\alpha,\beta)}(x), P_k^{(\alpha,\beta)}(x) \right>
\]

\[
= C_k \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta \left( P_k^{(\alpha,\beta)}(x) \right)^2 dx
\]

\[
= C_k \frac{2^{\alpha + \beta + 1}\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}{(2k + \alpha + \beta + 1)\Gamma(\alpha + \beta + k + 1)k!}.
\]

Therefore, by (1.7), (2.3), and (2.5), we obtain the following proposition.

**Proposition 2.1.** For \( q(x) \in P_n(n \in \mathbb{N}) \), one has

\[
q(x) = \sum_{k=0}^{n} C_k P_k^{(\alpha,\beta)}(x), \tag{2.6}
\]

where

\[
C_k = \frac{(-1)^k(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)}{2^{\alpha + \beta + 1}k!}\int_{-1}^{1} \left( \frac{d}{dx} \right)^k (1 - x)^{k+\alpha}(1 + x)^{k+\beta}q(x)dx. \tag{2.7}
\]

Let us take \( q(x) = x^n \in P_n \). First, we consider the following integral:

\[
\int_{-1}^{1} \left( \frac{d}{dx} \right)^k \left\{ (1 - x)^{k+\alpha}(1 + x)^{k+\beta} \right\} q(x)dx
\]

\[
= \int_{-1}^{1} \left( \frac{d}{dx} \right)^k \left\{ (1 - x)^{k+\alpha}(1 + x)^{k+\beta} \right\} x^ndx
\]

\[
= (-n) \int_{-1}^{1} \left( \frac{d}{dx} \right)^{k-1} \left\{ (1 - x)^{k+\alpha}(1 + x)^{k+\beta} \right\} x^{n-1}dx
\]

\[
= \ldots
\]

\[
= (-1)^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1 - x)^{k+\alpha}(1 + x)^{k+\beta}x^{n-k}dx
\]
By Proposition 2.1, we get
\[
\frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \int_0^1 y^{k+\beta} (1-y)^{k+\alpha} (2y-1)^{n-k} dy
= \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} B(k+l+\beta+1, k+\alpha+1)
= \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \frac{\Gamma(k+l+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+l+2)}.
\]
(2.8)

From (2.5) and (16), we have
\[
C_k = \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times \left[ \int_{-\infty}^{\infty} \left( \frac{d}{dx} \right)^k \left\{ (1-x)^{k+\alpha}(1+x)^{k+\beta} \right\} x^n dx \right]
= \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \frac{\Gamma(k+l+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+l+2)}.
\]
(2.9)

By Proposition 2.1, we get
\[
x^n = \frac{n!}{\Gamma(k+\beta+1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)(n-k)!} 2^k \times \left( \frac{(-1)^{n-k-l}(n-k) 2^l \Gamma(k+l+\beta+1)}{\Gamma(2k+\alpha+\beta+l+2)} \right) P_k^{(\alpha,\beta)}(x).
\]
(2.10)

From (1.9), we have
\[
e^{xt} = \frac{1}{t} \frac{t}{e^t - 1} e^{xt} (e^t - 1) = \sum_{n=0}^{\infty} \left( \frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1} \right) \frac{t^n}{n!}.
\]
(2.11)

By (2.11), we get
\[
x^n = \frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1}, \quad (n \in \mathbb{Z}_+).
\]
(2.12)

Therefore, by (2.10) and (2.12), we obtain the following theorem.
Theorem 2.2. For $n \in \mathbb{Z}_+$, one has

$$
\frac{1}{(n+1)!} \{B_{n+1}(x+1) - B_{n+1}(x)\}
= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{(-1)^{n-k-l}2^{k+l}(2k + \alpha + \beta + 1)(n-k)!}{\Gamma(k+\beta+1)\Gamma(2k + \alpha + \beta + l + 2)(n-k)!} \times \Gamma(k + \alpha + \beta + 1)\Gamma(k + l + \beta + 1)\binom{\alpha+\beta}{k}^r(x).
$$

(2.13)

Let us take $q(x) = B_n(x) \in \mathbb{P}_n$. Then we evaluate the following integral:

$$
\int_{-1}^{1} \left(\frac{d}{dx}\right)^k \left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\} B_n(x)dx
= \sum_{l=k}^{n} \binom{n}{l} B_{n-l} \frac{(-1)^{k}l!}{(l-k)!} 2^{k+\alpha+\beta+1} \int_{0}^{1} y^{k+\beta}(1-y)^{k+\alpha}(2y - 1)^{l-k} dy
= \sum_{l=k}^{n} \binom{n}{l} B_{n-l} \frac{(-1)^{k}l!}{(l-k)!} 2^{k+\alpha+\beta+1} \sum_{m=0}^{l-k} \binom{l-k}{m} 2^m (-1)^l (l-k-m)
\times \frac{\Gamma(k + m + \beta + 1)\Gamma(k + \alpha + 1)}{\Gamma(2k + \alpha + \beta + m + 2)}
= \sum_{l=k}^{n} \sum_{m=0}^{l-k} \frac{(-1)^{l-m}l!2^{k+\alpha+\beta+1}(l-k)!2^m\Gamma(k + m + \beta + 1)\Gamma(k + \alpha + 1)}{(l-k)!\Gamma(2k + \alpha + \beta + m + 2)}.

(2.14)

Finding (2.5) and (21), we have

$$
C_k = \frac{(-1)^{k}(2k + \alpha + \beta + 1)\Gamma(\alpha + \beta + k + 1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha + k + 1)\Gamma(\beta + k + 1)}
\times \int_{-1}^{1} \left(\frac{d}{dx}\right)^k \left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\} B_n(x)dx
= \sum_{l=k}^{n} \sum_{m=0}^{l-k} \frac{2^{k+m}(\binom{n}{l} B_{n-l})(-1)^{l-m-k}l!(2k + \alpha + \beta + 1)\binom{l-k}{m}}{\Gamma(\beta + k + 1)(l-k)!\Gamma(2k + \alpha + \beta + m + 2)}
\times \Gamma(k + m + \beta + 1)\Gamma(k + \alpha + \beta + 1).
$$

(2.15)
Theorem 2.3. For $n \in \mathbb{Z}_+$, one has

\[
B_n(x) = \sum_{k=0}^{n} \left( \sum_{l=k}^{n} \sum_{m=0}^{l-k} \frac{2^{k+m} \binom{k+m}{l-k} B_{n-l}(-1)^{l-m-k} l! (2k + \alpha + \beta + 1) \binom{l}{m}}{\Gamma(\beta + k + 1)(l-k)! \Gamma(2k + \alpha + \beta + m + 2)} \right) \times \Gamma(k + m + \beta + 1) \Gamma(k + \alpha + \beta + 1) P_{k}^{(\alpha,\beta)}(x).
\]

(2.16)

Let $q(x) = P_{n}^{(\alpha,\beta)}(x) \in \mathbb{P}_n$. From Proposition 2.1, we firstly evaluate the following integral:

\[
\int_{-1}^{1} \left( \frac{d}{dx} \right)^{k} [(1-x)^{k+\alpha}(1+x)^{k+\beta}] P_{n}^{(\alpha,\beta)}(x) dx
\]

\[
= (-1)^{k} \frac{1}{2^{k}} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(\alpha+k,\beta+k)}(x) dx.
\]

(2.17)

By (2.1) and (2.17), we get

\[
\int_{-1}^{1} \left( \frac{d}{dx} \right)^{k} [(1-x)^{k+\alpha}(1+x)^{k+\beta}] P_{n}^{(\alpha,\beta)}(x) dx
\]

\[
= (-1)^{k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{2^{k} \Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l} \binom{n+\beta}{l} \times \int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} \left( \frac{x-1}{2} \right)^{l} \left( \frac{x+1}{2} \right)^{n-k-l} dx
\]

\[
= (-1)^{k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{2^{k} \Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l} \binom{n+\beta}{l} \times \int_{0}^{1} (1-y)^{k+\alpha+l} y^{n+\beta-l} dy
\]

\[
= (-1)^{k} 2^{\alpha+k+1} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l} \binom{n+\beta}{l} (-1)^{l} \times B(k + \alpha + l + 1, n + \beta - l + 1)
\]

\[
= (-1)^{k} 2^{\alpha+k+1} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l} \binom{n+\beta}{l} (-1)^{l}
\]
It is easy to show that

\[
\frac{\Gamma(n + \beta - l + 1)}{\Gamma(\beta + k + 1)} = \frac{(n + \beta - l) \cdots \beta \Gamma(\beta)}{(\beta + k) \cdots \beta \Gamma(\beta)} = (n + \beta - l) \cdots (\beta + k + 1)
\]

\[
= \binom{n + \beta - l}{n - k - l} (n - k - l)!
\]

(2.19)

From (2.5), (2.18), and (2.19), we can derive the following equation:

\[
C_k = \frac{(\alpha + k + l + 1)\Gamma(n + \beta - l + 1)}{2\alpha x^{k+1}} \frac{\Gamma(\alpha + k + n + 2)}{\Gamma(\alpha + \beta + k + n + 2)}
\]

\[
\times \int_{-1}^{1} \left( \frac{d}{dx} \right)^k \left[ (1 - x)^{k+\alpha} (1 + x)^{k+\beta} \right] P_n^{(\alpha, \beta)}(x) dx
\]

\[
= \frac{(2k + \alpha + \beta + 1)\Gamma(\alpha + \beta + k + 1)}{\Gamma(\beta + k + 1)\Gamma(n + \alpha + \beta + 1)} \sum_{i=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} \binom{\alpha + k + l}{l}
\]

\[
\times l!(-1)^l \frac{\Gamma(n + \beta - l + 1)}{\alpha + \beta + k + n + 1}
\]

\[
= \frac{(2k + \alpha + \beta + 1)\Gamma(\alpha + \beta + k + 1)}{\alpha + \beta + k + n + 1} \sum_{i=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} \binom{n + \beta - l}{n - k - l}
\]

\[
\times \binom{n - k - l}{n - k - l}!! \frac{1}{\alpha + \beta + k + n + 1} (-1)^l.
\]

Therefore, by Proposition 2.1, we obtain the following theorem.
Theorem 2.4. For \((n \in \mathbb{Z}_+)\), one has

\[
\frac{\Gamma(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x)}{\Gamma(\alpha + \beta + 1)} = \sum_{k=0}^{n} \left\{ \sum_{l=0}^{n-k} (2k + \alpha + \beta + 1) \binom{\alpha + \beta + k}{k} \binom{n + \alpha}{n - k - l} \times \binom{n + \beta}{l} \binom{n + \beta - l}{l} \left( \frac{-1}{(n - k - l)!k!!} \right) \right\} P_k^{(\alpha, \beta)}(x).
\]

(2.21)

Let \(H_n(x)\) be the Hermite polynomial with

\[
H_n(x) = q(x) = \sum_{k=0}^{n} C_k P_k^{(\alpha, \beta)}(x),
\]

(2.22)

where

\[
C_k = \frac{(-1)^k (2k + \alpha + \beta + 1) \Gamma(k + \alpha + \beta + 1)}{2^k \Gamma(k + 1) \Gamma(\beta + k + 1)} \int_{-1}^{1} \left( \frac{d}{dx} \right)^k \left\{ (1 - x)^{k+\alpha} (1 + x)^{k+\beta} \right\} H_n(x) dx.
\]

(2.23)

Integrating by parts, one has

\[
\int_{-1}^{1} \left( \frac{d}{dx} \right)^k \left\{ (1 - x)^{k+\alpha} (1 + x)^{k+\beta} \right\} H_n(x) dx
\]

\[
= \frac{2^k (-1)^k n!}{(n-k)!} \int_{-1}^{1} (1 - x)^{k+\alpha} (1 + x)^{k+\beta} H_{n-k}(x) dx
\]

\[
= \frac{2^k (-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^l \int_{-1}^{1} (1 - x)^{k+\alpha} (1 + x)^{k+\beta} x^l dx
\]

\[
= \frac{2^k (-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^{k+\alpha+\beta+1} \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} 2^m
\]

(2.24)

\[
\times \int_{0}^{1} (1 - y)^{k+\alpha} y^{k+\beta+m} dy
\]

\[
= \frac{2^k (-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \sum_{m=0}^{l} \binom{n-k}{l} \binom{l}{m} H_{n-k-l} (-1)^{l-m} 2^{k+\alpha+\beta+m+l+1}
\]

\[
\times \frac{\Gamma(k + \alpha + 1) \Gamma(\beta + k + m + 1)}{\Gamma(2k + \alpha + \beta + m + 2)}.
\]
By (2.23) and (29), we get
\[
C_k = \sum_{l=0}^{n-k} \sum_{m=0}^l \binom{n-k}{l} \binom{l}{m} H_{n-k-l}(-1)^{l-m} \frac{(2k + \alpha + \beta + 1)(\alpha + \beta + k)k!}{(\alpha + \beta + 1)} \frac{(m + 2k)!}{(m+2k)}(m + 2k)! \cdot 2^{2k+m+l} \binom{\beta + k + m}{m} m!
\] (2.25)

Therefore, by (2.22) and (2.25), we obtain the following theorem.

**Theorem 2.5.** For \( n \in \mathbb{Z}_+ \), one has
\[
\frac{(\alpha + \beta + 1)H_n(x)}{n!} = \sum_{k=0}^{n} \left\{ \sum_{l=0}^{n-k} \sum_{m=0}^l \binom{n-k}{l} \binom{l}{m} H_{n-k-l}(-1)^{l-m} \frac{(2k + \alpha + \beta + 1)(\alpha + \beta + k)k!}{(\alpha + \beta + 1)} \frac{(m + 2k)!}{(m+2k)}(m + 2k)! \right. \nonumber \\
\left. \times \binom{\alpha + \beta + k}{k} \left( \binom{\beta + k + m}{m} m! \right) P_k^{(\alpha, \beta)}(x) \right\}
\] (2.26)

where \( H_n \) is the \( n \)th Hermite number.

**Remark 2.6.** By the same method as Theorem 2.3, we get
\[
\frac{1}{2n!} \left\{ E_n(x + 1) + E_n(x) \right\} = \sum_{k=0}^{n} \left( \sum_{l=0}^{n-k} \frac{2^{k+l}(2k + \alpha + \beta + 1)(\alpha + \beta + k)k!}{\Gamma(k + \beta + 1)\Gamma(2k + \alpha + \beta + l + 2)} \right. \nonumber \\
\left. \times \Gamma(k + \alpha + \beta + 1)\Gamma(k + l + \beta + 1) \right) P_k^{(\alpha, \beta)}(x).
\] (2.27)

**Acknowledgments**

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology 2012R1A1A2003786.

**References**


A. Bayad and T. Kim, “Identities involving values of Bernstein, \( q \)-Bernoulli, and \( q \)-Euler polynomials,” *Russian Journal of Mathematical Physics*, vol. 18, no. 2, pp. 133–143, 2011.


Submit your manuscripts at http://www.hindawi.com