Research Article

On the Max-Type Equation $x_{n+1} = \max\{1/x_n, A_n x_{n-1}\}$ with a Period-Two Parameter

İbrahim Yalçınkaya

Department of Mathematics, Ahmet Kelesoğlu Education Faculty, Konya University, 42090 Meram Campus, Meram Yeni Yol, Konya, Turkey

Correspondence should be addressed to İbrahim Yalçınkaya, iyalcinkaya1708@yahoo.com

Received 30 September 2011; Accepted 12 November 2011

Academic Editor: Cengiz Çinar

Copyright © 2012 İbrahim Yalçınkaya. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study the behavior of the well-defined solutions of the max type difference equation $x_{n+1} = \max\{1/x_n, A_n x_{n-1}\}$, $n = 0, 1, \ldots$, where the initial conditions are arbitrary nonzero real numbers and $\{A_n\}$ is a period-two sequence of real numbers with $A_n \in [0, \infty)$.

1. Introduction and Preliminaries

Recently, the study of max-type difference equations attracted a considerable attention. Although max-type difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the behavior of their solutions; see, for example [1–39] and the relevant references cited therein. Max-type difference equations stem from certain models in automatic control theory (see [1, 24]). For some papers on periodicity of difference equation, see, for example, [15, 16, 19, 22] and the relevant references cited therein.

In [9], Simsek et al. studied the behavior of the solutions of the following max-type difference equation:

$$x_{n+1} = \max\{x_{n-1}, 1/x_{n-1}\}, \quad n = 0, 1, \ldots,$$

where the initial conditions are nonzero real numbers.

In [10], Simsek studied the behavior of the solutions of the following max-type difference equation:

$$x_{n+1} = \max\{x_{n-2}, 1/x_{n-2}\}, \quad n = 0, 1, \ldots,$$
where the initial conditions are negative real numbers.

In [18], Elabbasy and Elsayed studied the behavior of the solutions of (1.2) where the initial conditions are nonzero real numbers.

In [20], Elsayed and Stević showed that every well-defined solution of the difference equation

\[ x_{n+1} = \max \left\{ \frac{A}{x_n}, x_{n-2} \right\}, \quad n = 0, 1, \ldots, \]

(1.3)

where \( A \in \mathbb{R} \), is eventually periodic with period three.

In [21], Elsayed and Iričanin showed that every positive solution to the following third-order nonautonomous max-type difference equation:

\[ x_{n+1} = \max \left\{ \frac{A_n}{x_n}, x_{n-2} \right\}, \quad n = 0, 1, \ldots, \]

(1.4)

where \( A_n \) is a three-periodic sequence of positive numbers and is periodic with period three.

In [29], Yalçinkaya et al. studied the behavior of the solutions of the following max-type difference equation:

\[ x_{n+1} = \max \left\{ \frac{1}{x_n}, A x_{n-1} \right\}, \quad n = 0, 1, \ldots, \]

(1.5)

where \( A \in \mathbb{R} \) and initial conditions are nonzero real numbers.

In this paper, we study the behavior of the well-defined solutions of the max type difference equation

\[ x_{n+1} = \max \left\{ \frac{1}{x_n}, A_n x_{n-1} \right\}, \quad n = 0, 1, \ldots, \]

(1.6)

where the initial conditions are arbitrary nonzero real numbers and \( \{A_n\} \) is a period-two sequence of real numbers with \( A_n \in [0, \infty) \).

We need the following definitions and lemmas.

**Definition 1.1.** A sequence \( \{x_n\}_{n=-k}^\infty \) is said to be eventually periodic with period \( p \) if there is \( n_0 \in \{-k, \ldots, -1, 0, 1, \ldots\} \) such that \( x_{n+p} = x_n \) for all \( n \geq n_0 \). If \( n_0 = -k \), then we say that the sequence \( \{x_n\}_{n=-k}^\infty \) periodic with period \( p \).

We make two definitions regarding (1.6).

**Definition 1.2.** A right semicycle is a string of terms \( x_l, \ldots, x_m \) with \( l \geq 1, m \leq \infty \) such that \( x_n = A_{n-1} x_{n-2} \) for all \( n = l, \ldots, m \). Furthermore, if \( l > 1 \), \( x_{l-1} = 1/x_{l-2} \), and if \( m < \infty \), \( x_{m+1} = 1/x_m \).

**Definition 1.3.** A left semicycle is a string of terms \( x_l, \ldots, x_m \) with \( l \geq 1, m \leq \infty \) such that \( x_n = 1/x_{n-1} \) for all \( n = l, \ldots, m \). Furthermore, if \( l > 1 \), \( x_{l-1} = A_{n-2} x_{n-3} \), and if \( m < \infty \), \( x_{m+1} = A_m x_{m-1} \).

We give the following lemmas which show us the periodic behavior of the solutions of (1.6).
Lemma 1.4. Assume that \( \{x_n\}_{n=0}^{\infty} \) is a well-defined solution of (1.6). If \( x_{n_0} = x_{n_0 + 2} \) and \( x_{n_0 + 1} = x_{n_0 + 3} \) such that \( n_0 \in \mathbb{N}_0 \cup \{-1\} \), then the solution \( \{x_n\}_{n=0}^{\infty} \) is eventually periodic with period two.

Proof. We prove that

\[ x_{n_0} = x_{n_0 + 2m}, \quad x_{n_0 + 1} = x_{n_0 + 2m + 1}, \quad (1.7) \]

by induction. For \( m = 1 \), this is, assumption. Assume that (1.7) holds for all \( 1 \leq n \leq m \). We may assume that \( n_0 \) is odd. Then, by the inductive hypothesis, we have

\[ x_{n_0 + 2(m_0 + 1)} = \max \left\{ \frac{1}{x_{n_0 + 2m_0 + 1}}, A_1 x_{n_0 + 2m_0 + 1} \right\} \]

\[ = \max \left\{ \frac{1}{x_{n_0 + 1}}, A_1 x_{n_0} \right\} = x_{n_0 + 1} = x_{n_0}, \quad (1.8) \]

from this and the inductive hypothesis, we have

\[ x_{n_0 + 2(m_0 + 1) + 2} = \max \left\{ \frac{1}{x_{n_0 + 2m_0 + 2}}, A_1 x_{n_0 + 2m_0 + 2} \right\} \]

\[ = \max \left\{ \frac{1}{x_{n_0}}, A_0 x_{n_0 + 1} \right\} = x_{n_0 + 3} = x_{n_0 + 1}, \quad (1.9) \]

which completes the proof (the case \( n_0 \) is even similar, so it will be omitted). \( \Box \)

We omit the proof of the following lemma, since it can easily be obtained by induction.

Lemma 1.5. Assume that \( \{x_n\}_{n=0}^{\infty} \) is a well-defined solution of (1.6). If \( x_{n_0}, x_{n_0 + 1} > 0 \) such that \( n_0 \in \mathbb{N}_0 \cup \{-1\} \), then \( x_n > 0 \) for all \( n \geq n_0 \).

Lemma 1.6. Assume that \( \{x_n\}_{n=0}^{\infty} \) is a well-defined solution of (1.6) and \( A_n \in [0, 1) \). If this solution is eventually positive, then it is eventually periodic with period two.

Proof. Assume that \( n_0 \in \mathbb{N}_0 \cup \{-1\} \) is the smallest index such that \( x_n > 0 \) for all \( n \geq n_0 \). Then, we have

\[ x_{n+1} x_n = \max \{1, A_n x_n x_{n-1} \} \quad \forall n \geq n_0 + 1. \quad (1.10) \]
Using this, we have
\[ x_{n_0+2} x_{n_0+1} = \max \{ 1, A_{n_0+1} x_{n_0+1} x_{n_0} \}, \]
\[ x_{n_0+3} x_{n_0+2} = \max \{ 1, A_{n_0+2} x_{n_0+2} x_{n_0+1} \} \]
\[ = \max \{ 1, A_{n_0+2} A_{n_0+1} x_{n_0+1} x_{n_0} \} \]
\[ = \max \{ 1, A_{n_0+2} A_{n_0+1} x_{n_0+1} x_{n_0} \}, \quad (1.11) \]
\[ x_{n_0+4} x_{n_0+3} = \max \{ 1, A_{n_0+3} A_{n_0+2} A_{n_0+1} x_{n_0+1} x_{n_0} \}, \]

then we get
\[ x_{n_0+k+1} x_{n_0+k} = \max \left\{ 1, x_{n_0+1} x_{n_0} \prod_{i=1}^{k} A_{n_0+i} \right\} \quad \forall k \geq 1. \quad (1.12) \]

Observe that there exists a positive integer \( k \) such that
\[ x_{n_0+1} x_{n_0} \prod_{i=1}^{k} A_{n_0+i} \leq 1. \quad (1.13) \]

From this directly follows that \( \{ x_n \}_{n=0}^{\infty} \) is eventually periodic with period two.

**Lemma 1.7.** Equation (1.6) has no right semicycle with an infinite terms for the positive initial conditions and \( 0 < A_0, A_1 < 1 \).

**Proof.** Conversely, assume that (1.6) has a right semicycle with an infinite terms. And, let \( \{ a_n \} \) be periodic sequence of natural numbers with period two such that \( \{ a_n \} = (0, 1, 0, 1, \ldots) \).

Without loss of generality, we denote by \( x_1 \) the first term of right semicycle with an infinite terms. There is at least \( n_0 \in \mathbb{N} \). For all \( n > n_0 \), we can write
\[ x_n = \max \left\{ A_0 \left[ (n/2) a_n A_{1} \frac{1}{x_{n-1}} x_{n-1} x_0 \right], A_0 \left[ ((n+1)/2) a_n A_{1} \frac{1}{x_{n-1}} x_{n-1} x_0 \right] \right\} \]
\[ = A_0 \left[ ((n+1)/2) a_n A_{1} \frac{1}{x_{n-1}} x_{n-1} x_0 \right], \quad (1.14) \]

which implies
\[ A_0 \left[ (n+1)/2 \right] a_n A_{1} \left[ ((n+1)/2) a_n \right] x_{n-1} x_0 > 1 \quad \forall n > n_0. \quad (1.15) \]

But this is a contradiction which completes the proof.

We omit the proof of the following lemma, since it can easily be obtained similarly.

**Lemma 1.8.** Equation (1.6) has no right semicycle with an infinite terms for the negative initial conditions and \( A_0, A_1 > 1 \).
2. **Main Results**

Since $A_n$ is a two periodic, it has the form $(A_0, A_1, A_0, A_1, \ldots)$. If $A_0 = A_1 = 0$, then (1.6) becomes $x_{n+1} = 1/x_n$, from which it follows that every well-defined solution is periodic with period two. Hence, in the sequel, we will consider the case when at least one of $A_0$ and $A_1$ is not zero.

2.1. **The Case** $0 < A_0, A_1 \leq 1$

**Theorem 2.1.** If $0 < A_0, A_1 \leq 1$ and at least one of the initial conditions is arbitrary positive real number, then every well-defined solution of (1.6) is eventually periodic with period two.

**Proof.** Firstly, assume that $x_{-1}, x_0 > 0$. Then, we have $x_1 = \max\{1/x_0, A_0 x_0\}$. There are two cases to be considered.

(a) If $A_0 x_{-1} x_0 \leq 1$, then $x_1 = 1/x_0$. Hence,

$$x_2 = \max\left\{\frac{1}{x_1}, A_1 x_0\right\} = \max\{x_0, A_1 x_0\} = x_0,$$

$$x_3 = \max\left\{\frac{1}{x_2}, A_2 x_1\right\} = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{1}{x_0}, \tag{2.1}\$$

$$x_4 = \max\left\{\frac{1}{x_3}, A_3 x_2\right\} = \max\{x_0, A_1 x_0\} = x_0.$$

From Lemma 1.4, the result follows.

(b) If $A_0 x_{-1} x_0 > 1$, then $x_1 = A_0 x_{-1}$. We have

$$x_2 = \max\left\{\frac{1}{x_1}, A_1 x_0\right\} = \max\left\{\frac{1}{A_0 x_{-1}}, A_1 x_0\right\}. \tag{2.2}\$$

There are two subcases to be considered.

(b1) If $A_0 A_1 x_{-1} x_0 \leq 1$, then $x_2 = 1/(A_0 x_{-1})$. Hence,

$$x_3 = \max\left\{\frac{1}{x_2}, A_2 x_1\right\} = \max\{A_0 x_{-1}, A_0^2 x_{-1}\} = A_0 x_{-1},$$

$$x_4 = \max\left\{\frac{1}{x_3}, A_3 x_2\right\} = \max\left\{\frac{1}{A_0 x_{-1}}, \frac{A_1}{A_0 x_{-1}}\right\} = \frac{1}{A_0 x_{-1}}. \tag{2.3}\$$

From Lemma 1.4, the result follows in this case.

(b2) If $A_0 A_1 x_{-1} x_0 > 1$, then $x_2 = A_1 x_0$. We have

$$x_3 = \max\left\{\frac{1}{x_2}, A_2 x_1\right\} = \max\left\{\frac{1}{A_1 x_0}, A_0^2 x_{-1}\right\}. \tag{2.4}\$$
There are two subcases to be considered.

(b21) If $A_0^2 A_1 x_{-1} x_0 \leq 1$, then $x_3 = 1/(A_1 x_0)$. We have

\[
\begin{align*}
\quad x_4 &= \max \left\{ \frac{1}{x_3}, A_3 x_2 \right\} = \max \left\{ A_1 x_0, A_1^2 x_0 \right\} = A_1 x_0, \\
\quad x_5 &= \max \left\{ \frac{1}{A_1 x_0}, A_4 x_3 \right\} = \max \left\{ \frac{1}{A_1 x_0}, \frac{A_0}{A_1 x_0} \right\} = \frac{1}{A_1 x_0}.
\end{align*}
\] (2.5)

From Lemma 1.4, the result follows in this case.

(b22) If $A_0^2 A_1 x_{-1} x_0 > 1$, then $x_3 = A_0^2 x_{-1}$. The result follows Lemma 1.7.

Secondly, assume that $x_0 < 0 < x_{-1}$, then we have

\[
\begin{align*}
\quad x_1 &= \max \left\{ \frac{1}{x_0}, A_0 x_{-1} \right\} = A_0 x_{-1} > 0, \\
\quad x_2 &= \max \left\{ \frac{1}{x_1}, A_1 x_0 \right\} = \frac{1}{x_1} > 0.
\end{align*}
\] (2.6)

From Lemmas 1.5 and 1.6, the result follows (the case $x_{-1} < 0 < x_0$ is similar, so it will be omitted) which completes the proof.

Remark 2.2. If $0 < A_0, A_1 \leq 1$ and $x_{-1}, x_0 < 0$, then every well-defined solution of (1.6) is not periodic.

2.2. The Case $A_0 = 0 < A_1 < 1$ or $A_1 = 0 < A_0 < 1$

Theorem 2.3. If $A_0 = 0 < A_1 < 1$ or $A_1 = 0 < A_0 < 1$, then every well-defined solution of (1.6) is eventually periodic with period two.

Proof. First assume that $A_1 = 0 < A_0 < 1$. Then, we have

\[
\begin{align*}
\quad x_1 &= \max \left\{ \frac{1}{x_0}, A_0 x_{-1} \right\} > 0, \\
\quad x_2 &= \max \left\{ \frac{1}{x_1}, A_1 x_0 \right\} > 0.
\end{align*}
\] (2.7)

From Lemmas 1.5 and 1.6, the result follows. The case $A_0 = 0 < A_1 < 1$ is similar, so it will be omitted.

2.3. The Other Cases

If at least one of $A_0$ and $A_1$ greater than one, then we have the well-defined solutions of (1.6), where the positive initial conditions are not periodic. So, there are many cases in which solutions of (1.6) are not periodic. If the solutions of (1.6) are not periodic, then general solution of (1.6) can be obtained for many subcases.
Theorem 2.4. Assume that \( \{ x_n \}_{n=1}^{\infty} \) is a well-defined solution of (1.6) for \( A_0 > 1 \) and \( A_1 = 0 \).

(a) If \( x_1 = 1/x_0 \) and \( x_0, x_{-1} > 0 \) or \( x_{-1} < 0 < x_0 \), then
\[
x_n = \left( \frac{x_0}{A_0^{\lfloor n/2 \rfloor}} \right)^{(-1)^n}.
\] (2.8)

(b) If \( x_1 = A_0 x_{-1} \) and \( x_0, x_{-1} > 0 \) or \( x_0 < 0 < x_{-1} \), then
\[
x_n = \left( \frac{1}{A_0^{\lfloor (n+1)/2 \rfloor}} \right)^{(-1)^n}.
\] (2.9)

Proof. (a) It can be proved by induction. Let \( x_1 = 1/x_0 \) and \( x_0, x_{-1} > 0 \). For \( n = 1 \), (2.8) holds. Assume that (2.8) holds for all \( 1 \leq m \leq m_0 \). We may assume that \( m_0 \in \mathbb{N} \) is even (the case \( m_0 \) is odd is similar, so it will be omitted). Then, by the inductive hypothesis, we have
\[
x_{m_0+1} = \max \left\{ \frac{1}{x_{m_0}}, A_{m_0} x_{m_0-1} \right\}
\]
\[
= \max \left\{ \frac{A_0^{m_0-2}/2}{x_0}, \frac{A_{m_0}^{m_0}/2}{x_0} \right\} = \frac{A_0^{m_0}/2}{x_0} = \left( \frac{x_0}{A_0^{\lfloor m_0/2 \rfloor}} \right)^{(-1)^{m_0+1}},
\] (2.10)
which completes the proof.

(b) Also, this case can be proved similarly. \( \square \)

Now, we describe the behavior of solutions of (1.6) for some other cases. We omit the proof of the following theorem, since it can easily be obtained by induction.

Theorem 2.5. Assume that \( \{ x_n \}_{n=-1}^{\infty} \) is a well-defined solution of (1.6).

(a) If \( A_0 = 0, A_1 > 1 \) and \( x_0, x_{-1} > 0 \) (or \( x_{-1} < 0 < x_0 \)), then
\[
x_n = \left( A_1^{\lfloor n/2 \rfloor} x_0 \right)^{(-1)^n}.
\] (2.11)

(b) If \( A_0, A_1 > 1 \) and \( x_0, x_{-1} > 0 \) (or \( x_{-1} < 0 < x_0, x_1 = 1/x_0 \)), then
\[
x_n = \left( A_0^{a_{n-1}} A_1^{a_n} \right)^{\lfloor n/2 \rfloor} x_0^{(-1)^n}.
\] (2.12)

(c) If \( A_0, A_1 > 1 \) and \( x_0, x_{-1} > 0, x_1 = A_0 x_{-1}, x_2 = A_1 x_0 \), then
\[
x_n = \left( A_0^{\lfloor (n+1)/2 \rfloor a_{n-1}} A_1^{\lfloor (n+1)/2 \rfloor a_n} \right)^{\lfloor n/2 \rfloor} x_0^{a_n} x_{-1}^{a_{n-1}}.
\] (2.13)

There are many different cases. The different cases can be obtained similarly.
Theorem 2.6. If \( A_0, A_1 > 1 \) and initial conditions are negative, then every well-defined solution of (1.6) is eventually periodic with period two.

Proof. Assume that \( x_0, x_{-1} < 0 \). Then,

\[
x_1 = \max \left\{ \frac{1}{x_0}, A_0 x_{-1} \right\}.
\] (2.14)

There are two cases to be considered.

(a) If \( x_1 = 1/x_0 \), then \( x_2 = x_0, x_3 = 1/x_0, x_4 = x_0 \). Then, the result follows Lemma 1.4.

(b) If \( x_1 = A_0 x_{-1} \), then \( x_2 = \max \{1/(A_0 x_{-1}), A_1 x_0\} \). There are two subcases.

(b1) If \( x_2 = 1/(A_0 x_{-1}) \), then \( x_3 = A_0 x_{-1}, x_4 = 1/(A_0 x_{-1}) \). Then the result follows Lemma 1.4.

(b2) If \( x_2 = A_0 x_{-1} \), then there will be subcases and from Lemmas 1.4 and 1.8 which completes the proof.

\[\Box\]

Acknowledgment

I am grateful to the anonymous referees for their valuable suggestions that improved the quality of this study.

References


[31] R. Abu-Saris and F. Allan, “Periodic and nonperiodic solutions of the difference equation \(x_{n+1} = \max\{x_{n}^{2}, A\}/x_{n}x_{n-1}\),” in *Advances in Di erence Equations*, pp. 9–17, Gordon and Breach, Amsterdam, The Netherlands, 1997.


[33] S. Stević, “On the recursive sequence \(x_{n+1} = \max\{c, x_{n}^{2}/x_{n-1}^{p}\}\),” *Applied Mathematics Letters*, vol. 21, no. 8, pp. 791–796, 2008.

[34] T. Sun, B. Qin, H. Xi, and C. Han, “Global behavior of the max-type difference equation \(x_{n+1} = \max\{1/x_{n}, A/n, x_{n-1}\}\),” *Abstract and Applied Analysis*, vol. 2009, Article ID 152964, 10 pages, 2009.

Submit your manuscripts at
http://www.hindawi.com