Research Article

Integral Formulae of Bernoulli Polynomials

Dae San Kim, Dmitry V. Dolgy, Hyun-Mee Kim, Sang-Hun Lee, and Taekyun Kim

1 Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
2 Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea
3 Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea
4 Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

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Recently, some interesting and new identities are introduced in [1]. From these identities, we derive some new and interesting integral formulae for the Bernoulli polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by generating functions as follows:

\[ \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.1) \]

(see [1–11]). In the special case, \( x = 0 \), \( B_n(0) = B_n \) are called the \( n \)th Bernoulli numbers. The Euler polynomials are also defined by

\[ \frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.2) \]

with the usual convention about replacing \( E^n(x) \) by \( E_n(x) \) (see [1–11]). From (1.1) and (1.2),
we can easily derive the following equation:

\[
\frac{t}{e^t - 1} e^{xt} = \frac{t}{2} \left( \frac{2e^{xt}}{e^t + 1} \right) + \left( \frac{t}{e^t - 1} \right) \left( \frac{2e^{xt}}{e^t + 1} \right)
\]

\[= \sum_{n=0}^\infty \left( \sum_{k=0}^n \frac{n!}{k!} B_k E_{n-k} (x) \right) \frac{t^n}{n!}.
\]

By (1.1) and (1.3), we get

\[B_n (x) = \sum_{k=0}^n \binom{n}{k} B_k E_{n-k} (x), \quad (n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}). \tag{1.4}\]

From (1.1), we have

\[B_n (x) = \sum_{l=0}^n \binom{n}{l} B_l x^{n-l}. \tag{1.5}\]

Thus, by (1.5), we get

\[\frac{d}{dx} B_n (x) = n \sum_{l=0}^{n-1} \binom{n-1}{l} B_l x^{n-1-l} = nB_{n-1} (x). \tag{1.6}\]

It is known that \(E_n (0) = E_n\) are called the \(n\)th Euler numbers (see [7]). The Euler polynomials are also given by

\[E_n (x) = \sum_{l=0}^n \binom{n}{l} E_l x^{n-l}, \tag{1.7}\]

(see [6]). From (1.7), we can derive the following equation:

\[\frac{d}{dx} E_n (x) = n \sum_{l=0}^{n-1} \binom{n}{l} E_l x^{n-1-l} = nE_{n-1} (x). \tag{1.8}\]

By the definition of Bernoulli and Euler numbers, we get the following recurrence formulae:

\[E_0 = 1, \quad E_n (1) + E_n = 2\delta_{0,n}, \quad B_0 = 1, \quad B_n (1) - B_n = \delta_{1,n}, \tag{1.9}\]

where \(\delta_{n,k}\) is the kronecker symbol (see [5]). From (1.6), (1.8), and (1.9), we note that

\[\int_0^1 B_n (x) dx = \frac{\delta_{0,n}}{n+1}, \quad \int_0^1 E_n (x) dx = -\frac{2E_{n+1}}{n+1}. \tag{1.10}\]
where \( n \in \mathbb{Z}_+. \) The following identity is known in [5]:

\[
\sum_{j=0}^{m} \sum_{l=0}^{k} (a + b + 1)^{m-j}(a + 1)^{k-l} \binom{m}{j} \binom{k}{l} \frac{(-1)^{j+l}}{j + l + 1}
+ \sum_{j=0}^{m} \sum_{l=0}^{k} ((a + b + 1)^{m-j}(a + 1)^{k-l} - (a + b)^{m-j}a^{k-l}) \binom{m}{j} \binom{k}{l} \frac{B_{j+l+1}(x)}{j + l + 1}
= (x + a)^k (x + a + b)^m,
\]

where \( a, b \in \mathbb{Z}. \)

From the identities of Bernoulli polynomials, we derive some new and interesting integral formulae of an arithmetical nature on the Bernoulli polynomials.

### 2. Integral Formulae of Bernoulli Polynomials

From (1.1) and (1.2), we note that

\[
\frac{2}{e^t + 1} e^{xt} = \frac{1}{t} \left( \frac{2(e^t - 1)}{e^t + 1} \right) \left( \frac{t e^{xt}}{e^t - 1} \right)
= \frac{1}{t} \left( 2 - 2 \frac{2}{e^t + 1} \right) \left( \frac{t e^{xt}}{e^t - 1} \right)
= \frac{1}{t} \left( - \sum_{l=1}^{\infty} \frac{2E_l t^l}{l!} \right) \left( \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \right)
= -2 \left( \sum_{l=0}^{\infty} \frac{E_{l+1} t^l}{l + 1 l!} \right) \left( \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \right)
= -2 \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{E_{l+1}}{l + 1 l!} \binom{n}{l} B_{n-l}(x) \right) \frac{t^n}{n!}.
\]

Therefore, by (1.2) and (2.1), we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}_+, \) one has

\[
E_n(x) = -2 \sum_{l=0}^{n} \left( \frac{n}{l + 1} \right) E_{l+1} B_{n-l}(x).
\] (2.2)

Let us take the definite integral from 0 to 1 on both sides of (1.4): for \( n \geq 2, \)

\[
0 = -2 \sum_{k=0}^{n} \binom{n}{k} B_k \frac{E_{n-k+1}}{n - k + 1} = -2B_n E_1 - 2 \sum_{k=1}^{n-1} \binom{n}{k} B_k \frac{E_{n-k+1}}{n - k + 1}.
\] (2.3)
By (2.3), we get

$$B_n = 2 \sum_{k=0}^{n-1} \binom{n}{k} B_{k} E_{n-k+1}.$$  \hfill (2.4)

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.2.** For $n \in \mathbb{N}$, with $n \geq 2$, one has

$$B_n = 2 \sum_{k=0}^{n-1} \binom{n}{k} B_{k} E_{n-k+1}.$$  \hfill (2.5)

Let us take $k = m$, $a = 0$, and $b = -2$ in (1.11). Then we have

$$\sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} (-1)^{j+l} \binom{m}{j} \binom{m}{l} B_{j+l+1}(x) + \sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} B_{j+l+1}(x) = x^m(x-2)^m.$$  \hfill (2.6)

It is easy to show that

$$\int_{0}^{1} x^m(x-2)^m dx = 2 \int_{0}^{1/2} (2t-2)^m(2t)^m dt$$

$$= (-1)^m 2^{2m} \left( 2 \int_{0}^{1/2} t^m(1-t)^m dt \right) = (-1)^m 2^{2m} \frac{1}{(2m+1)(\binom{2m}{m})}.$$  \hfill (2.7)

Let us consider the integral from 0 to 1 in (2.6):

$$\sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} \frac{1}{j+l+1} = \frac{(-1)^m 2^{2m}}{(2m+1)(\binom{2m}{m})} (m \in \mathbb{N}).$$  \hfill (2.8)
By (2.6) and (2.8), we get

\[
\sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} \frac{B_{j+l+1}}{j+l+1} = 2 \sum_{j=0}^{m} (-2)^{m-j} \binom{m}{j} \frac{1}{j+m+1} \sum_{k=1}^{j+m} \frac{(j+m+1)}{k} \frac{B_k E_{j+m+2-k}}{j+m+2-k} + \frac{(-1)^{m+1} 2^{2m}}{2m+1} \frac{1}{\left(\frac{2m}{m}\right)},
\]

for \( m \in \mathbb{N} \).

Therefore, by (2.9), we obtain the following theorem.

**Theorem 2.3.** For \( m \in \mathbb{N} \), one has

\[
\sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} B_{j+l+1} = 2 \sum_{j=0}^{m} (-2)^{m-j} \binom{m}{j} \frac{1}{j+m+1} \sum_{k=1}^{j+m} \frac{(j+m+1)}{k} \frac{B_k E_{j+m+2-k}}{j+m+2-k} + \frac{(-1)^{m+1} 2^{2m}}{2m+1} \frac{1}{\left(\frac{2m}{m}\right)}.
\]

**Lemma 2.4.** Let \( a, b \in \mathbb{Z} \). For \( m, k \in \mathbb{Z}_+ \), one has

\[
\sum_{j=0}^{m} \sum_{l=0}^{k} (a+b+1)^{m-j}(a+1)^{k-l} \binom{m}{j} \binom{k}{l} E_{j+l}(x) = 2(a+b)^m(x+a)^k,
\]

(see [5]).

Let us take \( k = m, a = 1, b = -2 \) in Lemma 2.4. Then we have

\[
\sum_{l=0}^{m} 2^{m-l} \binom{m}{l} E_{m+l}(x) + \sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} E_{j+l}(x) = 2(2 - x)^m.
\]

Taking integral from 0 to 1 in (2.12), we get

\[
-2 \sum_{l=0}^{m} 2^{m-l} \binom{m}{l} E_{m+l+1} \frac{1}{m+l+1} - 2 \sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{m}{l} E_{j+l+1} \frac{1}{j+l+1} = 2 \int_{0}^{1} (x^2 - 1)^m dx.
\]
It is easy to show that
\[ \int_0^1 (x^2 - 1)^m \, dx = (-1)^m \prod_{k=1}^{m} \left( \frac{2k}{2k+1} \right) = \frac{(-1)^m 2^m}{(2m+1) \binom{2m}{m}}. \] (2.14)

Thus, by (2.13) and (2.14), we get
\[ \sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{j} \binom{m}{l} E_{j+l+1} = \sum_{l=0}^{m} 2^{m-l} \binom{m}{l} E_{m+l+1} + \frac{(-1)^{m+1} 2^m}{(2m+1) \binom{2m}{m}}. \] (2.15)

Therefore, by (2.2) and (2.15), we obtain the following theorem.

**Theorem 2.5.** For \( m \in \mathbb{Z}_+ \), one has
\[ \sum_{j=0}^{m} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{j} \binom{m}{l} E_{j+l+1} = \sum_{l=0}^{m} 2^{m-l} \binom{m}{l} E_{m+l+1} + \frac{(-1)^{m+1} 2^m}{(2m+1) \binom{2m}{m}}. \] (2.16)

### 3. \( p \)-Adic Integral on \( \mathbb{Z}_p \) Associated with Bernoulli and Euler Numbers

Let \( p \) be a fixed odd prime number. Throughout this section, \( \mathbb{Z}_p, \mathbb{Q}_p, \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( v_p \) be the normalized exponential valuation of \( \mathbb{C}_p \), with \( |p|_p = p^{-v_p(p)} = 1/p \). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the bosonic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by
\[ I(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x), \] (3.1)
(see [8]). Thus, by (3.1), we get
\[ \int_{\mathbb{Z}_p} f_1(x) \, d\mu(x) = \int_{\mathbb{Z}_p} f(x) \, d\mu(x) + f'(0), \] (3.2)
where \( f_1(x) = f(x+1) \), and \( f'(0) = df(x)/dx|_{x=0} \). Let us take \( f(y) = e^{t(x+y)} \). Then we have
\[ \int_{\mathbb{Z}_p} e^{t(x+y)} \, d\mu(y) = \frac{t}{e^t-1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \] (3.3)
From (3.3), we have

\[ \int_{\mathbb{Z}_p} (x + y)^n d\mu(y) = B_n(x), \quad \int_{\mathbb{Z}_p} y^n d\mu(y) = B_n. \]  

(3.4)

From (1.2), we can derive the following integral equation:

\[ I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \quad (n \in \mathbb{N}). \]  

(3.5)

Thus, from (3.4) and (3.5), we get

\[ \int_{\mathbb{Z}_p} (x + n)^m d\mu(x) = \int_{\mathbb{Z}_p} x^m d\mu(x) + m \sum_{i=0}^{n-1} i^{m-1}. \]  

(3.6)

From (3.6), we have

\[ B_m(n) - B_m = m \sum_{i=0}^{n-1} i^{m-1} \quad (m \in \mathbb{Z}_+). \]  

(3.7)

The fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by Kim as follows [6, 7]:

\[ I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \to \infty} \sum_{x=0}^{n^{-1}} f(x)(-1)^x. \]  

(3.8)

Let \( f_1(x) = f(x + 1) \). Then we have

\[ I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \]

\[ I_{-1}(f_2) = -I_{-1}(f_1) + 2f_1(0) = -I_{-1}(f_1) + 2f(1) \]  

(3.9)

Continuing this process, we obtain the following equation:

\[ I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l), \quad \text{where } f_n(x) = f(x + n). \]  

(3.10)

Thus, by (3.10), we have

\[ \int_{\mathbb{Z}_p} (x + n)^m d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} l^m. \]  

(3.11)
Let us take \( f(y) = e^{t(x+y)} \). By (3.9), we get

\[
\int_{Z_p} e^{t(x+y)} \, d\mu(y) = \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\] (3.12)

From (3.2), we have the Witt’s formula for the \( n \)th Euler polynomials and numbers as follows:

\[
\int_{Z_p} (x+y)^n \, d\mu(y) = E_n(x), \quad \int_{Z_p} y^n \, d\mu(y) = E_n, \quad \text{where } n \in \mathbb{Z}_+.
\] (3.13)

By (3.11) and (3.13), we get

\[
E_m(n) = (-1)^n \left( E_m + 2 \sum_{l=0}^{n-1} (-1)^{l-1} l^m \right), \quad (m \in \mathbb{Z}_+, n \in \mathbb{N}).
\] (3.14)

Let us consider the following \( p \)-adic integral on \( Z_p \):

\[
K_1 = \int_{Z_p} B_n(x) \, d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{Z_p} x^l \, d\mu(x)
\] (3.15)

\[
= \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l.
\]

From (1.4) and (3.15), we have

\[
K_1 = \sum_{k,l \leq n \land k \neq l} \binom{n}{k} B_k \sum_{l=0}^{n-k} \binom{n-k}{l} \left( \begin{array}{c} n-k \\; l \\ \end{array} \right) \int_{Z_p} x^l \, d\mu(x)
\] (3.16)

\[
= \sum_{k,l \leq n \land k \neq l} \binom{n-k}{k} \binom{n}{l} \left( \begin{array}{c} n-k \; l \\ \end{array} \right) B_k B_l E_{n-k-l}.
\]

Therefore, by (3.15) and (3.16), we obtain the following theorem.

**Theorem 3.1.** For \( n \in \mathbb{Z}_+ \), one has

\[
\sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l = \sum_{k,l \leq n \land k \neq l} \binom{n-k}{k} \binom{n}{l} \left( \begin{array}{c} n-k \; l \\ \end{array} \right) B_k B_l E_{n-k-l}.
\] (3.17)

Now, we set

\[
K_2 = \int_{Z_p} B_n(x) \, d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} E_l.
\] (3.18)
By (1.4), we get

\[
K_2 = \sum_{k=0}^{n} \binom{n}{k} B_k \sum_{l=0}^{n-k} E_{n-k-l} \binom{n-k}{l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x)
\]

\[
= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n-k}{l} B_k E_{n-k-l}.
\]

(3.19)

Therefore, by (3.18) and (3.19), we obtain the following theorem.

**Theorem 3.2.** For \( n \in \mathbb{Z}_+ \), one has

\[
\sum_{l=0}^{n} \binom{n}{l} B_n E_l = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n-k}{l} B_k E_{n-k-l}.
\]

(3.20)

Let us consider the following integral on \( \mathbb{Z}_p \):

\[
K_3 = \int_{\mathbb{Z}_p} E_n(x) d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} E_l.
\]

(3.21)

From (2.2), we have

\[
K_3 = -2 \sum_{l=0}^{n} \binom{n}{l} \sum_{k=0}^{n-l} \binom{n-l}{k} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x)
\]

\[
= -2 \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} B_{n-l-k} E_{l+1} / (l+1) E_{n-l-k} E_k.
\]

(3.22)

Therefore, by (3.21) and (3.22), we obtain the following theorem.

**Theorem 3.3.** For \( n \in \mathbb{Z}_+ \), one has

\[
\sum_{l=0}^{n} \binom{n}{l} E_{n-l} E_l = -2 \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n-l}{k} B_{n-l-k} E_k.
\]

(3.23)

Now, we set

\[
K_4 = \int_{\mathbb{Z}_p} E_n(x) d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} E_l.
\]

(3.24)
By (2.2), we get

\[ K_4 = -2 \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \frac{E_{l+1}}{l+1} B_{n-l-k} B_k. \]  

(3.25)

Therefore, by (3.24) and (3.25), we obtain the following corollary.

**Corollary 3.4.** For \( n \in \mathbb{Z}_+ \), we have

\[ \sum_{l=0}^{n} \binom{n}{l} E_l B_l = -2 \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \frac{E_{l+1}}{l+1} B_{n-l-k} B_k. \]  

(3.26)

Let us assume that \( a, b, c, d \in \mathbb{Z} \). From Lemma 2.4 and (3.13), we note that

\[
\int_{z_p} ((a + b + 1) + (x + y))^m ((a + 1) + (x + y))^k d\mu_{l-1}(y) \\
+ \int_{z_p} ((a + b) + (x + y))^m ((a + (x + y))^k d\mu_{l-1}(y) \\
= 2(x + a + b)^m (x + a)^k.
\]  

(3.27)

By (3.27), we get

\[
2(x + a + b)^m (x + a)^k = \int_{z_p} ((a + b - c + 1) + (x + c + y))^m ((a - c + 1) + (x + c + y))^k d\mu_{l-1}(y) \\
+ \int_{z_p} ((a + b - d) + (x + y + d))^m ((a - d) + (x + y + d))^k d\mu_{l-1}(y).
\]  

(3.28)

Thus, by (3.28) and (3.13), we obtain the following lemma (see [5]).

**Lemma 3.5.** Let \( a, b, c, d \in \mathbb{Z} \). For \( m, k \in \mathbb{Z}_+ \), one has

\[
\sum_{j=0}^{m} \sum_{l=0}^{k} \binom{m}{j} \binom{k}{l} (a + b - c + 1)^{m-j} (a - c + 1)^{k-l} E_{j+l}(x + c) \\
+ \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - d)^{m-j} (a - d)^{k-l} \binom{m}{j} \binom{k}{l} E_{j+l}(x + d) = 2(x + a + b)^m (x + a)^k.
\]  

(3.29)
Let us consider the formula in Lemma 3.5 with \( d = c - 1 \). Then we have

\[
\sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c + 1)^{m-j}(a - c + 1)^{k-l} \binom{m}{j} \binom{k}{l} (E_{j+l}(x+c) + E_{j+l}(x + c - 1))
= 2(x + a + b)^m (x + a)^k.
\]

Taking \( \int_{\mathbb{Z}_p} d\mu(x) \) on both sides of (3.30),

\[
\text{LHS} = 2 \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c + 1)^{m-j}(a - c + 1)^{k-l} \binom{m}{j} \binom{k}{l} \sum_{s=0}^{j+l} \binom{j+l}{s} E_{j+l-s} \int_{\mathbb{Z}_p} ((x + c)^s + (x + c - 1)^s) d\mu(x)
\]

\[
= 2 \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c + 1)^{m-j}(a - c + 1)^{k-l} \binom{m}{j} \binom{k}{l} \sum_{s=0}^{j+l} \binom{j+l}{s} E_{j+l-s} B_s(c-1) + \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c + 1)^{m-j}(a - c + 1)^{k-l} \binom{m}{j} \binom{k}{l} (j+l) E_{j+l-1}(c-1).
\]

By the same method, we get

\[
\text{RHS} = 2 \sum_{s=0}^{m} \binom{m}{s} b^{m-s} \int_{\mathbb{Z}_p} (x + a)^{s+k} d\mu(x)
= 2 \sum_{s=0}^{m} \binom{m}{s} b^{m-s} B_{s+k}(a).
\]

Therefore, by (3.31) and (3.32), we obtain the following proposition.

**Proposition 3.6.** Let \( a, b, c \in \mathbb{Z} \). Then one has

\[
2 \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c + 1)^{m-j}(a - c + 1)^{k-l} \binom{m}{j} \binom{k}{l} \sum_{s=0}^{j+l} \binom{j+l}{s} E_{j+l-s} B_s(c-1) + \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c + 1)^{m-j}(a - c + 1)^{k-l} \binom{m}{j} \binom{k}{l} (j+l) E_{j+l-1}(c-1)
= 2 \sum_{s=0}^{m} \binom{m}{s} b^{m-s} B_{s+k}(a).
\]
Replacing $c$ by $c + 1$, we have

$$2 \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c)^{m-j}(a - c)^{k-l} \left(\binom{m}{j} \binom{k}{l} \sum_{s=0}^{j+l} \binom{j+l}{s} \right) E_{j+l-s} B_s(c)$$

$$+ \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c)^{m-j}(a - c)^{k-l} \left(\binom{m}{j} \binom{k}{l} \right) E_{j+l-s}(c)$$

$$= 2 \sum_{s=0}^{m} \binom{m}{s} b^{m-s} B_{s+k}(a).$$

From (3.4) and (3.7), we derive some identity for the first term of the LHS of (3.34). The first term of the LHS of (3.34)

$$= 2 \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c)^{m-j}(a - c)^{k-l} \left(\binom{m}{j} \binom{k}{l} \sum_{s=0}^{j+l} \binom{j+l}{s} \right) E_{j+l-s}$$

$$\times \left( B_s + s \sum_{i=0}^{c-1} t^{i-1} \right)$$

$$= 2 \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c)^{m-j}(a - c)^{k-l} \left(\binom{m}{j} \binom{k}{l} \sum_{s=0}^{j+l} \binom{j+l}{s} \right) E_{j+l-s} B_s$$

$$+ 2 \sum_{i=0}^{c-1} \sum_{j=0}^{m} \sum_{l=0}^{k} (a + b - c)^{m-j}(a - c)^{k-l} \left(\binom{m}{j} \binom{k}{l} \right) (j+l)(-1)^0$$

$$\times \left( E_{j+l-1} + 2 \sum_{e=0}^{i-1} (-1)^{i-1} e^{j+l-1} \right)$$

$$= 2 \sum_{j=0}^{m} \sum_{l=0}^{j+l} \sum_{s=0}^{\min\{j+l,m\}} \left(\binom{m}{j} \binom{k}{l} \binom{j+l}{s} \right) (a + b - c)^{m-j}(a - c)^{k-l}$$

$$\times E_{j+l-s} B_s + 2m \sum_{j=0}^{m-1} \sum_{l=0}^{k} \binom{m-1}{j} \binom{k}{l} (a + b - c)^{m-1-j}(a - c)^{k-l} E_{j+l} \delta_{c+1}$$

$$+ 2k \sum_{j=0}^{m-1} \sum_{l=0}^{k-1} \binom{m}{j} \binom{k-1}{l} (a + b - c)^{m-j}(a - c)^{k-1-l} E_{j+l} \delta_{c+1}$$

$$+ 4m \sum_{e=0}^{c-2} (a + b - c + e)^{m-1}(a - c + e)^k \delta_{c+e}$$

$$+ 4k \sum_{e=0}^{c-2} (a + b - c + e)^m(a - c + e)^{k-1} \delta_{c+e}. $$
where

\[
\delta_{c=k} = \begin{cases} 
1 & \text{if } c \equiv k \mod 2, \\
0 & \text{if } c \not\equiv k \mod 2.
\end{cases}
\]  

(3.36)

The second term of the LHS of (3.34)

\[
= \sum_{j=0}^{m} \sum_{l=0}^{k} (j + l)(a + b - c)^{m-j}(a - c)^{k-l} \binom{m}{j} \binom{k}{l} (-1)^{j} \left( E_{j+l-1} + 2 \sum_{i=0}^{c-1} (-1)^{i-j} i^{j-l-1} \right)
\]

\[
\equiv \sum_{j=0}^{m} \sum_{l=0}^{k} (j + l)(a + b - c)^{m-j}(a - c)^{k-l} \binom{m}{j} \binom{k}{l} E_{j+l-1}
\]

\[
+ 2(-1)^{c} \sum_{i=0}^{c-1} (-1)^{i-j} m(a + b - c + i)^{m-j} (a - c + i)^{k-l-1}
\]

\[
+ 2(-1)^{c} \sum_{i=0}^{c-1} (-1)^{i-j} (a + b - c + i)^{m} k(a - c + i)^{k-1}
\]  

(3.37)

Therefore, by (3.34), (3.35), and (3.37), we obtain the following theorem.

Theorem 3.7. Let $a, b, c \in \mathbb{Z}$ with $c \geq 1$. Then one has

\[
2 \sum_{j=0}^{m} \sum_{l=0}^{k} \sum_{s=0}^{j+l}(\binom{m}{j}) \binom{k}{l} \binom{j + l}{s} (a + b - c)^{m-j}(a - c)^{k-l} E_{j+l-s} B_s
\]

\[
+ 2m \delta_{c=1} \sum_{j=0}^{m} \sum_{l=0}^{k} \binom{m-1}{j} \binom{k}{l} (a + b - c)^{m-1-j}(a - c)^{k-l} E_{j+l}
\]

\[
+ 2k \delta_{c=1} \sum_{j=0}^{m} \sum_{l=0}^{k} \binom{m}{j} \binom{k-1}{l} (a + b - c)^{m-j}(a - c)^{k-1-l} E_{j+l}
\]

\[
+ (-1)^{c} \sum_{j=0}^{m} \sum_{l=0}^{k} (j + l)(a + b - c)^{m-j}(a - c)^{k-l} \binom{m}{j} \binom{k}{l} E_{j+l-1}
\]
\[ + 2m \sum_{e=0}^{c-1} (a + b - c + e)^{m-1} (a - c + e)^k \]
\[ + 2k \sum_{e=0}^{c-1} (a + b - c + e)^m (a - c + e)^k \]
\[ = 2m \sum_{s=0}^{c} \binom{m}{s} b^{m-s} B_{s+k}(a), \]
\[ (3.38) \]

where

\[ \delta_{c=k} = \begin{cases} 
1 & \text{if } c \equiv k \pmod{2}, \\
0 & \text{if } c \not\equiv k \pmod{2}.
\end{cases} \]
\[ (3.39) \]

**Remark 3.8.** Here, we note that

\[ 4m \sum_{e=0}^{c-2} (a + b - c + e)^{m-1} (a - c + e)^k \delta_{c=k} + 2m(-1)^c \sum_{i=0}^{c-1} (-1)^i (a + b - c + i)^{m-1} (a - c + i)^k \]
\[ = 2m \sum_{e=0}^{c-2} (a + b - c + e)^{m-1} (a - c + e)^k. \]
\[ (3.40) \]

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**References**


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