

Research Article

q-Analogues of the Bernoulli and Genocchi Polynomials and the Srivastava-Pintér Addition Theorems

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Received 24 April 2012; Revised 5 July 2012; Accepted 23 July 2012

Academic Editor: Lee Chae Jang

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The main purpose of this paper is to introduce and investigate a new class of generalized Bernoulli and Genocchi polynomials based on the q-integers. The q-analogues of well-known formulas are derived. The q-analogue of the Srivastava-Pintér addition theorem is obtained.

1. Introduction

Throughout this paper, we always make use of the following notation: \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{N}_0 \) denotes the set of nonnegative integers, \( \mathbb{R} \) denotes the set of real numbers, and \( \mathbb{C} \) denotes the set of complex numbers.

The q-shifted factorial is defined by

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N}, \quad (a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - q^j a), \quad |q| < 1, \quad a \in \mathbb{C}.
\] (1.1)

The q-numbers and q-numbers factorial is defined by

\[
[a]_q = \frac{1 - q^a}{1 - q}; \quad [0]_q = 1; \quad [n]_q! = [1]_q [2]_q \cdots [n]_q, \quad n \in \mathbb{N}, \quad a \in \mathbb{C},
\] (1.2)
respectively. The $q$-polynomial coefficient is defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q; q)_n}{(q; q)_{n-k}(q; q)_k},$$

(1.3)

The $q$-analogue of the function $(x + y)^n$ is defined by

$$(x + y)_q^n := \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (q^{1/2}k(k-1)x^{n-k}y^k), \quad n \in \mathbb{N}_0.$$  

(1.4)

In the standard approach to the $q$-calculus two exponential function are used:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q)q^k z)}, \quad 0 < |q| < 1, \ |z| < \frac{1}{|1 - q|},$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{(1/2)n(n-1)}z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left( 1 + (1 - q)q^k z \right), \quad 0 < |q| < 1, \ z \in \mathbb{C}.$$  

(1.5)

From this form we easily see that $e_q(z)E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \quad D_q E_q(z) = E_q(qz),$$

(1.6)

where $D_q$ is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}.$$  

(1.7)

The previous $q$-standard notation can be found in [1].

Carlitz has introduced the $q$-Bernoulli numbers and polynomials in [2]. Srivastava and Pinté proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [3]. They also gave some generalizations of these polynomials. In [4–6], Kim et al. investigated some properties of the $q$-Euler polynomials and Genocchi polynomials. They gave some recurrence relations. In [7], Cenkci et al. gave the $q$-extension of Genocchi numbers and polynomials. In [5], Kim gave a new concept for the $q$-Genocchi numbers and polynomials. In [8], Simsek et al. investigated the $q$-Genocchi zeta function and $l$-function by using generating functions and Mellin transformation. We also recall the definitions of the $q$-Bernoulli and the $q$-Genocchi polynomials of higher order (see [2, 9–12]):

$$(-t)^n \sum_{n=0}^{\infty} \left[ \begin{array}{c} [n]_q \\ n \end{array} \right] \frac{q^{n+x} e_{\left[ n+x \right]} t^n}{n!} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!},$$

(1.8)

$$2t^n \sum_{n=0}^{\infty} \left[ \begin{array}{c} [n]_q \\ n \end{array} \right] (-1)^n q^{n+x} e_{\left[ n+x \right]} t^n = \sum_{n=0}^{\infty} C_{n,q}(x) \frac{t^n}{n!}.$$
We propose the following definitions. We define the \( q \)-Bernoulli and the \( q \)-Genocchi polynomials of higher order in two variables \( x \) and \( y \), using two \( q \)-exponential functions, which helps us easily prove some properties of these polynomials and \( q \)-analogue of the Srivastava and Pintér addition theorem.

**Definition 1.1.** The \( q \)-Bernoulli numbers \( \mathcal{B}_{n,q}^{(a)} \) and polynomials \( \mathcal{B}_{n,q}^{(a)}(x,y) \) in \( x,y \) of order \( \alpha \) are defined by means of the generating function functions:

\[
\left( \frac{t}{e_q(t) - 1} \right)^\alpha = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(a)} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,
\]

\[
\left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi.
\]  

**Definition 1.2.** The \( q \)-Genocchi numbers \( \mathcal{G}_{n,q}^{(a)} \) and polynomials \( \mathcal{G}_{n,q}^{(a)}(x,y) \) in \( x,y \) are defined by means of the generating functions:

\[
\left( \frac{2t}{e_q(t) + 1} \right)^\alpha = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(a)} \frac{t^n}{[n]_q!}, \quad |t| < \pi,
\]

\[
\left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.
\]

It is obvious that

\[
\mathcal{B}_{n,q}^{(a)} = \mathcal{B}_{n,q}^{(a)}(0,0), \quad \lim_{q \to 1} \mathcal{B}_{n,q}^{(a)}(x,y) = B_n^{(a)}(x+y), \quad \lim_{q \to 1} \mathcal{B}_{n,q}^{(a)} = B_n^{(a)},
\]

\[
\mathcal{G}_{n,q}^{(a)} = \mathcal{G}_{n,q}^{(a)}(0,0), \quad \lim_{q \to 1} \mathcal{G}_{n,q}^{(a)}(x,y) = G_n^{(a)}(x+y), \quad \lim_{q \to 1} \mathcal{G}_{n,q}^{(a)} = G_n^{(a)}.
\]  

Here \( B_n^{(a)}(x) \) and \( E_n^{(a)}(x) \) denote the classical Bernoulli, and Genocchi polynomials of order \( \alpha \) are defined by

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{tx} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!}, \quad \left( \frac{2}{e^t + 1} \right)^\alpha e^{tx} = \sum_{n=0}^{\infty} G_n^{(a)}(x) \frac{t^n}{n!}.
\]  

The aim of the present paper is to obtain some results for the \( q \)-Genocchi polynomials (properties of the \( q \)-Bernoulli polynomials are studied in [13]). The \( q \)-anallogues of well-known results, for example, Srivastava and Pintér [3], can be derived from these \( q \)-identities. It should be mentioned that probabilistic proofs the Srivastava-Pintér addition theorems were given recently in [14]. The formulas involving the \( q \)-Stirling numbers of the second kind, \( q \)-Bernoulli polynomials and \( q \)-Bernstein polynomials, are also given. Furthermore some special cases are also considered.

The following elementary properties of the \( q \)-Genocchi polynomials \( \mathcal{G}_{n,q}^{(a)}(x,y) \) of order \( \alpha \) are readily derived from Definition 1.2. We choose to omit the details involved.
Property 1.3. Special values of the $q$-Genocchi polynomials of order $\alpha$:

$$\mathcal{E}^{(0)}_{n,q}(x,0) = x^n, \quad \mathcal{E}^{(0)}_{n,q}(0,y) = q^{(1/2)n(n-1)}y^n. \quad (1.13)$$

Property 1.4. Summation formulas for the $q$-Genocchi polynomials of order $\alpha$:

$$\mathcal{E}^{(a)}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{E}^{(a)}_{k,q}(x+y)^{n-k}, \quad \mathcal{E}^{(a)}_{n,q}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{E}^{(a)}_{n-k,q}(x,y),$$

$$\mathcal{G}^{(a)}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathcal{G}^{(a)}_{k,q}(x,0) y^{n-k}, \quad \mathcal{G}^{(a)}_{n,q}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathcal{G}^{(a)}_{k,q} y^{n-k}. \quad (1.14)$$

Property 1.5. Difference equations:

$$\mathcal{G}^{(a)}_{n,q}(1,y) + \mathcal{G}^{(a)}_{n,q}(0,y) = 2[n]_q \mathcal{G}^{(a-1)}_{n-1,q}(0,y),$$

$$\mathcal{G}^{(a)}_{n,q}(x,0) + \mathcal{G}^{(a)}_{n,q}(x,-1) = 2[n]_q \mathcal{G}^{(a-1)}_{n-1,q}(x,-1). \quad (1.15)$$

Property 1.6. Differential relations:

$$D_{q,x} \mathcal{G}^{(a)}_{n,q}(x,y) = [n]_q \mathcal{G}^{(a)}_{n-1,q}(x,y), \quad D_{q,y} \mathcal{G}^{(a)}_{n,q}(x,y) = [n]_q \mathcal{G}^{(a)}_{n-1,q}(x,qy). \quad (1.16)$$

Property 1.7. Addition theorem of the argument:

$$\mathcal{E}^{(a+b)}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{E}^{(a)}_{n-k,q}(x,0) \mathcal{E}^{(b)}_{k,q}(0,y). \quad (1.17)$$

Property 1.8. Recurrence relationships:

$$\mathcal{E}^{(a)}_{n,q}(\frac{1}{m},y) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{1}{m} - 1\right)^{-k} \mathcal{E}^{(a)}_{k,q}(0,y) = 2[n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(\frac{1}{m} - 1\right)^{-k} \mathcal{E}^{(a-1)}_{k,q}(0,y). \quad (1.18)$$

2. Explicit Relationship between the $q$-Genocchi and the $q$-Bernoulli Polynomials

In this section we prove an interesting relationship between the $q$-Genocchi polynomials $\mathcal{G}^{(a)}_{n,q}(x,y)$ of order $\alpha$ and the $q$-Bernoulli polynomials. Here some $q$-analogues of known results will be given. We also obtain new formulas and their some special cases in the following.
Theorem 2.1. For \( n \in \mathbb{N}_0 \), the following relationship

\[
\mathcal{G}^{(a)}_{n,q}(x, y) = \sum_{k=0}^{n} \frac{1}{m^{n-k-1}[k+1]_q} \left[ 2[k + 1]_q \sum_{j=0}^{k} \frac{1}{m^{k-j}} \mathcal{G}^{(a-1)}_{j,q}(x,-1) - \sum_{j=0}^{k+1} \frac{1}{m^{k+1-j}} \mathcal{G}^{(a)}_{j,q}(x,-1) - \mathcal{G}^{(a)}_{k+1,q}(x,0) \right] \mathcal{B}_{n-k,q}(0, my)
\]

holds true between the \( q \)-Genocchi and the \( q \)-Bernoulli polynomials.

Proof. Using the following identity:

\[
\left( \frac{2t}{e_q(t) + 1} \right)^a e_q(tx)E_q(ty) = \left( \frac{2t}{e_q(t) + 1} \right)^a e_q(tx) \cdot e_q(t/m - 1) - e_q(t/m - 1) \cdot E_q(t/m my),
\]

we have

\[
\sum_{n=0}^{\infty} \mathcal{G}^{(a)}_{n,q}(x, y) t^n \frac{[n]_q}{[n]_q!} = \sum_{n=0}^{m} \sum_{k=0}^{n} \frac{1}{m^{n-k}} \mathcal{G}^{(a)}_{k,q}(x,0) \cdot e_q(t/m - 1) - e_q(t/m - 1) \cdot E_q(t/m my)
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{1}{m^{n-k}} \mathcal{G}^{(a)}_{k,q}(x,0) t^n \frac{[n]_q}{[n]_q!} - m \mathcal{G}^{(a)}_{n,q}(x,0) t^n \frac{[n]_q}{[n]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(0, my) t^n \frac{[n]_q}{[n]_q!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{1}{m^{n+1}} \mathcal{G}^{(a)}_{n+1,q}(x,0) t^n \frac{[n+1]_q}{[n+1]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(0, my) t^n \frac{[n]_q}{[n]_q!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{m^{n+k+1}} \mathcal{G}^{(a)}_{k+1,q}(x,0) \left( \sum_{j=0}^{k+1} \frac{1}{m^{k+1-j}} \mathcal{G}^{(a)}_{j,q}(x,0) - \mathcal{G}^{(a)}_{k+1,q}(x,0) \right) \mathcal{B}_{n-k,q}(0, my) t^n \frac{[n]_q}{[n]_q!}.
\]

It remains to use Property 1.8. \( \square \)
Since $G_n^{(a)}(x, y)$ is not symmetric with respect to $x$ and $y$, we can prove a different form of the previously mentioned theorem. It should be stressed out that Theorems 2.1 and 2.2 coincide in the limiting case when $q \to 1$.\\

**Theorem 2.2.** For $n \in \mathbb{N}_0$, the following relationship

$$
G_n(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{1}{m^{n-k-1}} \left[ k+1 \right]_q \left[ 2[k+1] \sum_{j=0}^{k} \left[ \begin{array}{c} k \\ j \end{array} \right]_q \left( \frac{1}{m} - 1 \right)^{j-k} G_{j,q}(0, y)  \\
- \sum_{j=0}^{k+1} \left[ k+1 \\ j \end{array} \right]_q \left( \frac{1}{m} - 1 \right)^{k+1-j} G_{j,q}(0, y) - G_{k+1,q}(0, y) \right] \\
	imes \mathcal{B}_{n-k,q}(mx, 0)
$$

(2.4)

holds true between the $q$-Genocchi and the $q$-Bernoulli polynomials.\\

**Proof.** The proof is based on the following identity:

$$
\left( \frac{2t}{e_q(t)+1} \right)^{a} e_q(t x) E_q(t y) = \left( \frac{2t}{e_q(t)+1} \right)^{a} E_q(t y) \cdot \frac{e_q(t/m) - 1}{t} \cdot \frac{t}{e_q(t/m) - 1} \cdot e_q \left( \frac{t}{m} mx \right).
$$

(2.5)

Next we discuss some special cases of Theorems 2.1 and 2.2. By noting that

$$
G_{i,q}^{(0)}(0, y) = q^{(1/2)(j-1)} y^i, \quad G_{i,q}^{(0)}(x, -1) = (x - 1)^{j}_q,
$$

(2.6)

we deduce from Theorems 2.1 and 2.2 Corollary 2.3 below.

**Corollary 2.3.** For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship

$$
G_n(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{1}{m^{n-k-1}} \left[ k+1 \right]_q \left[ 2[k+1] \sum_{j=0}^{k} \left[ \begin{array}{c} k \\ j \end{array} \right]_q \left( \frac{1}{m} - 1 \right)^{j-k} q^{(1/2)(j-1)} y^i  \\
- \sum_{j=0}^{k+1} \left[ k+1 \\ j \end{array} \right]_q \left( \frac{1}{m} - 1 \right)^{k+1-j} \mathcal{B}_{j,q}(0, y) - \mathcal{B}_{k+1,q}(0, y) \right] \\
	imes \mathcal{B}_{n-k,q}(mx, 0),
$$
\[ \mathcal{G}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^{n-k-1}[k+1]_q} \left[ 2[k+1]_q \sum_{j=0}^{k} \binom{k}{j} \frac{1}{m^{k-j}} (x-1)^j \right] \]

\[ \times \mathcal{B}_{n-k,q}(0, my) \]

(2.7)

holds true between the \( q \)-Bernoulli polynomials and \( q \)-Euler polynomials.

**Corollary 2.4.** For \( n \in \mathbb{N}_0, m \in \mathbb{N} \) the following relationship holds true:

\[ G_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \frac{2}{k+1} ((k+1)y^k - G_{k+1,q}(y)) B_{n-k}(x), \]

(2.8)

\[ G_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^{n-k-1}(k+1)} \left[ 2(k+1)G_k \left( y + \frac{1}{m} - 1 \right) \right. \]

\[ \left. -G_{k+1} \left( y + \frac{1}{m} - 1 \right) - G_{k+1}(y) \right] B_{n-k,q}(mx) \]

(2.9)

between the classical Genocchi polynomials and the classical Bernoulli polynomials.

Note that the formula (2.9) is new for the classical polynomials.

In terms of the \( q \)-Genocchi numbers \( \mathcal{G}_{n,q}^{(a)} \), by setting \( y = 0 \) in Theorem 2.1, we obtain the following explicit relationship between the \( q \)-Genocchi polynomials \( \mathcal{G}_{k,q}^{(a)} \) of order \( \alpha \) and the \( q \)-Bernoulli polynomials.

**Corollary 2.5.** The following relationship holds true:

\[ \mathcal{G}_{n,q}^{(a)}(x, 0) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^{n-k-1}[k+1]_q} \left[ 2[k+1]_q \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{m} - 1 \right)^{k-j} \mathcal{G}_{j,q}^{(a-1)} \right. \]

\[ \left. -\sum_{j=0}^{k+1} \binom{k+1}{j} \left( \frac{1}{m} - 1 \right)^{k+1-j} \mathcal{G}_{j,q}^{(a)} - \mathcal{G}_{k+1,q}^{(a)} \right] \mathcal{B}_{n-k,q}(mx, 0). \]

(2.10)

**Corollary 2.6.** For \( n \in \mathbb{N}_0 \) the following relationship holds true:

\[ \mathcal{G}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{2}{[k+1]_q} \left[ (k+1)q^{(1/2)k(k-1)} y^k - \mathcal{G}_{k+1,q}(0, y) \right] \mathcal{B}_{n-k,q}(x, 0). \]

(2.11)
Corollary 2.7. For $n \in \mathbb{N}_0$ the following relationship holds true:

$$G_{n,q}(x,0) = -\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \frac{2}{[k+1]_q} \mathcal{G}_{k+1,q} B_{n-k,q}(x,0),$$

$$G_{n,q} = -\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \frac{2}{[k+1]_q} \mathcal{G}_{k+1,q} B_{n-k,q}.$$ 

(2.12)

References

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