Partial Extinction, Permanence, and Global Attractivity in Nonautonomous $n$-Species Gilpin-Ayala Competitive Systems with Impulses

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The qualitative properties of general nonautonomous $n$-species Gilpin-Ayala competitive systems with impulsive effects are studied. Some new criteria on the permanence, extinction, and global attractivity of partial species are established by using the methods of inequalities estimate and Liapunov functions.

1. Introduction

In [1], the general nonautonomous $n$-species Lotka-Volterra competitive systems with impulsive effects are investigated. By using the methods of inequalities estimate and constructing the suitable Liapunov functions, the sufficient conditions on the permanence of whole species and global attractivity of systems are established.

In [2], the authors studied the following general nonautonomous $n$-species Lotka-Volterra competitive systems with impulsive perturbations:

\[
\begin{align*}
\dot{x}_i(t) &= x_i(t) \left[ a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t) \right], \quad t \neq t_k, \\
x_i(t^+_k) &= h_{ik}x_i(t_k), \quad k = 1, 2, \ldots, i = 1, 2, \ldots, n,
\end{align*}
\]

and got a series of criteria on the extinction of a part of $n$-species, the permanence of other part of $n$-species, and the global attractivity of the systems.
In [3], a periodic $n$-species Gilpin-Ayala competition system with impulses is studied and obtain some useful behaviors of the system.

In this paper, we investigate the general nonautonomous $n$-species Gilpin-Ayala competitive systems with impulsive effects.

\[
\dot{x}_i(t) = x_i(t) \left[ a_i(t) - \sum_{j=1}^{n} b_{ij}(t) x_j^\theta_{ij}(t) \right], \quad t \neq t_k,
\]

where \( b_i(t) \) and \( a_{ij}(t) \) \((i, j = 1, 2, \ldots, n)\) are defined on \( \mathbb{R}_+ = [0, \infty) \) and are bounded continuous functions, \( a_{ij}(t) \geq 0 \) for all \( t \in \mathbb{R}_+ \), \( \theta_{ij} \) and \( h_{ik} > 0 \) are constants for all \( k = 1, 2, \ldots \) and \( i, j = 1, 2, \ldots, n \).

2. Preliminaries

Firstly, we introduce the following assumption.

**Assumption H.** There is a positive constant \( \omega \) such that for each \( i = 1, 2, \ldots, n \)

\[
\liminf_{t \to \infty} \int_{t}^{t+\omega} a_{ii}(s) ds > 0, \quad \liminf_{t \to \infty} \left( \int_{t}^{t+\omega} b_{i}(s) ds + \sum_{t \leq t_k \leq t+\mu} \ln h_{ik} \right) > 0,
\]

and functions

\[
h_i(t, \mu) = \sum_{t \leq t_k \leq t+\mu} \ln h_{ik}, \quad i = 1, 2, \ldots, n
\]

are bounded on \( t \in \mathbb{R}_+ \), and \( 0 \leq \mu \leq \omega \).

For each \( i \in \{1, 2, \ldots, n\} \), we consider the following logistic impulsive equation as the subsystem of system (1.2)

\[
\dot{x}_i(t) = x_i(t) \left[ b_i(t) - a_{ii}(t) x_i^{\theta_{ii}}(t) \right], \quad t \neq t_k,
\]

where

\[
x_i(t_k^+) = h_{ik} x_i(t_k), \quad k = 1, 2, \ldots
\]

From the above assumption, we have the following results.
Lemma 2.1. Suppose that assumption H holds. Then we have the following:

(1) There exist positive constants $m$ and $M$ such that

$$m \leq \liminf_{t \to \infty} x_i(t) \leq \limsup_{t \to \infty} x_i(t) \leq M,$$

for any positive solution $u_i(t)$ of (2.3).

(2) $\lim_{t \to \infty} (x_i^{(1)}(t) - x_i^{(2)}(t)) = 0$ for any two positive solutions $x_i^{(1)}(t)$ and $x_i^{(2)}(t)$ of (2.3).

Proof. From assumption H, there are positive constants $k_1, k_2, \delta$ and $T_0$ such that for all $t \geq T_0$ we have

\[\int_t^{t+\omega} (b_i(s) - a_{ii}(s)k_1)ds + \sum_{t \leq s < t+\omega} \ln h_k < -\delta,\]

\[\int_t^{t+\omega} (b_i(s) - a_{ii}(s)k_2)ds + \sum_{t \leq s < t+\omega} \ln h_k > \delta.\]

From the boundedness of function $h(t, \mu) = \sum_{t \leq s < t+\omega} \ln h_k$, there is a positive constant $P$ such that for any $t \in R_+$ and $\mu \in [0, \omega)$

$$|h(t, \mu)| = \left| \sum_{t \leq s < t+\mu} \ln h_k \right| < P. \tag{2.7}$$

Firstly, we prove that there is a constant $M > 0$ such that

$$\limsup_{t \to \infty} x_i(t) < M, \tag{2.8}$$

for any positive solution $x_i(t)$ of system (2.3). In fact, for any positive solution $x_i(t)$ of system (2.3), we only need to consider the following three cases.

Case I. There is a $t_0 \geq T_0$ such that $x(t) \geq k_1'$ for all $t \geq t_0$.

Case II. There is a $t_0 \geq T_0$ such that $x(t) \leq k_1'$ for all $t \geq t_0$.

Case III. $x(t)$ is oscillatory about $k_1'$ for all $t \geq T_0$.\]
We first consider Case I. Since \( x_i(t) \geq k'_1 \) for all \( t \geq t_0 \), then for \( t = t_0 + l\omega \), where \( l \geq 0 \) is any positive integer, integrating system (2.3) from \( t_0 \) to \( t \), from (2.5) we have

\[
x_i(t) = x_i(t_0) \exp \left( \int_{t_0}^{t} \left( b_i(s) - a_{ii}(s)x_i^0(s) \right) ds + \sum_{t_0 \leq \tau < l} \ln h_k \right)
\]

\[
\leq x_i(t_0) \exp \left( \int_{t_0}^{t_{0} + l_0 \omega} (b_i(s) - a_{ii}(s)k_1) ds + \sum_{t_0 \leq \tau < l_0} \ln h_k + \cdots \right) + \int_{t_0 + (l-1)\omega}^{t_{0} + l_0 \omega} (b_i(s) - a_{ii}(s)k_1) ds + \sum_{t_0 + (l-1)\omega \leq \tau < t_0 + l_0 \omega} \ln h_k \\
\leq x_i(t_0) \exp (-l\delta).
\]

Hence, \( x_i(t) \to 0 \) as \( l \to \infty \), which leads a contradiction.

Next, we consider Case III. From the oscillation of \( x_i(t) \) about \( k'_1 \), we can choose two sequences \( \{ \rho_n \} \) and \( \{ \rho^*_n \} \) satisfying \( T_0 < \rho_1 < \rho^*_1 < \cdots < \rho_n < \rho^*_n < \cdots \) and \( \lim_{n \to \infty} \rho_n = \lim_{n \to \infty} \rho^*_n = \infty \) such that

\[
x_i(\rho_n) \leq k'_1, \quad x_i(\rho^*_n) \geq k'_1, \quad x_i(\rho^*_n) \geq k'_1, \quad x_i(\rho^*_n) \leq k'_1, \\
x_i(t) \geq k'_1, \quad \forall t \in (\rho_n, \rho^*_n), \\
x_i(t) \leq k'_1, \quad \forall t \in (\rho^*_n, \rho_{n+1}).
\]

For any \( t \geq T_0 \), if \( t \in (\rho_n, \rho^*_n) \) for some integer \( n \), then we can choose integer \( l \geq 0 \) and constant \( 0 \leq \mu_1 < \omega \) such that \( t = \rho_n + l\omega + \mu_1 \). Since

\[
\dot{x}_i(t) \leq x_i(t) (b_i(t) - a_{ii}(t)k_1), \quad \forall t \in (\rho_n, \rho^*_n), \ t \neq t_k,
\]
integrating this inequality from \( \rho_n \) to \( t \), by (2.5) and (2.7) we obtain

\[
x_i(t) = x_i(\rho_n) \exp \left( \int_{\rho_n}^{t} \left( b_i(s) - a_{ii}(s)x^{\rho_n}(s) \right) ds + \sum_{\rho_n \leq k < t} \ln h_k \right)
\]

\[
\leq k'_1 \exp \left( \int_{\rho_n}^{\rho_n + \omega} (b_i(s) - a_{ii}(s)k_1) ds + \sum_{\rho_n \leq k < \rho_n + \omega} \ln h_k + \cdots + \int_{\rho_n + \omega + \mu_1}^{\rho_n + 2\omega + \mu_1} (b_i(s) - a_{ii}(s)k_1) ds + \sum_{\rho_n + \omega + \mu_1 \leq k < \rho_n + 2\omega + \mu_1} \ln h_k \right)
\]

\[
\leq k'_1 \exp \left( -l_\delta + \int_{\rho_n + \omega + \mu_1}^{\rho_n + 2\omega + \mu_1} (b_i(s) - a_{ii}(s)k_1) ds + \sum_{\rho_n + \omega + \mu_1 \leq k < \rho_n + 2\omega + \mu_1} \ln h_k \right)
\]

where \( \alpha_1 = \sup_{t \in [0, 1]} \|b_i(t)\|_+ + a_{ii}(t)k_1 \). If there is an integer \( n \) such that \( t \in (\rho_n, \rho_{n+1}) \), then we obviously have

\[
x_i(t) \leq k'_1 < k'_1 \exp(\alpha_1 \omega + P).
\]

Therefore, for Case III we always have

\[
x_i(t) \leq k'_1 \exp(\alpha_1 \omega + P), \quad \forall t \geq T_0.
\]

Lastly, if Case II holds, then we directly have

\[
x_i(t) \leq k'_1 \exp(\alpha_1 \omega + P), \quad \forall t \geq T_0.
\]

Choose constant \( M = k'_1 \exp(\alpha_1 \omega + P) \), then we see that (2.8) holds.

Secondly, a similar argument as in the proof of (2.8) we can prove that there is a constant \( m > 0 \), such that

\[
\liminf_{t \to \infty} x(t) > m,
\]

for any positive solution \( x_i(t) \) of system (2.3). Conclusion (1.1) is proved.

Now, we prove conclusion (1.2). Let \( x_i^{(1)}(t) \) and \( x_i^{(2)}(t) \) be any two positive solutions of system (2.3). From conclusion (1.1), it follows that there are positive constants \( A \) and \( B \) such that

\[
A \leq x_i^{(1)}(t), \quad x_i^{(2)}(t) \leq B, \quad \forall t \geq 0.
\]
Choose Lyapunov function as follows:

\[ V(t) = \left| \ln x_i^{(1)}(t) - \ln x_i^{(2)}(t) \right|. \] (2.18)

For any \( k = 1, 2, \ldots \), we have

\[ V(t_k^+) = \left| \ln \left( h_k x_i^{(1)}(t_k) \right) - \ln \left( h_k x_i^{(2)}(t_k) \right) \right| = V(t_k). \] (2.19)

Hence, \( V(t) \) is continuous for all \( t \in R \) and from the Mean-Value Theorem we can obtain

\[ \frac{1}{B} \left| x_i^{(1)}(t) - x_i^{(2)}(t) \right| \leq V(t) \leq \frac{1}{A} \left| x_i^{(1)}(t) - x_i^{(2)}(t) \right|. \] (2.20)

Calculating the upper right derivative of \( V(t) \), then from (2.20) we obtain

\[
D^+ V = \text{sign} \left( x_i^{(1)}(t) - x_i^{(2)}(t) \right) \left( \frac{x_i^{(1)}(t)}{x_i^{(1)}(t)} - \frac{x_i^{(2)}(t)}{x_i^{(2)}(t)} \right)
\]

\[
= -a_{ii}(t) \left| x_i^{(1)k_i}(t) - x_i^{(2)k_i}(t) \right|
\]

\[
\leq -a_{ii}(t) \left[ \theta_{ii} \right] A_{ii} \left| x_i^{(1)}(t) - x_i^{(2)}(t) \right|
\]

\[
\leq -a_{ii}(t) \left[ \theta_{ii} \right] A_{ii} V(t), \quad t \neq t_k, \quad k = 1, 2, \ldots,
\]

where \( \left[ \theta_{ii} \right] \leq \theta_{ii} \) is the integer part of \( \theta_{ii} \).

From this, we further have for any \( t > 0 \)

\[
V(t) \leq V(0) \exp \left( -\left[ \theta_{ii} \right] A_{ii} \int_0^t a_{ii}(s) ds \right).
\] (2.22)

From condition (2.5) we can obtain \( \int_0^t a_{ii}(t) dt \to \infty \) as \( t \to \infty \). Hence, \( V(t) \to 0 \) as \( t \to \infty \).

Further from (2.20) we finally obtain \( \lim_{t \to \infty} (x_i^{(1)}(t) - x_i^{(2)}(t)) = 0 \). Conclusion (1.2) is proved.

This completes the proof of Lemma 2.1.

Applying Lemma 2.1 and the comparison theorem of impulsive differential equations, we easily prove the following result.

**Lemma 2.2.** Suppose that assumption \( H \) holds then there is a constant \( B \geq 0 \) such that

\[
\lim \sup_{t \to \infty} x_i(t) \leq B, \quad i = 1, 2, \ldots, n,
\] (2.23)

for any positive solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1.2).
3. Extinction

On the partial extinction of system (1.2), we have the following result.

Theorem 3.1. Suppose that assumption H holds. Let \( r \) be a given integer and \( 1 \leq r < n \). If for any \( l > r \) there is a \( i_l < l \) such that for any \( j \leq l \)

\[
\theta_{ij} = \theta_{l_j}, \tag{3.1}
\]

\[
\limsup_{t \to \infty} \frac{\int_t^{t+\omega} b_i(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{ik}}{\int_t^{t+\omega} b_i(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{ik}} < \liminf_{t \to \infty} \frac{a_i(t)}{a_{ij}(t)}, \quad \forall j \leq l, \tag{3.2}
\]

or

\[
\liminf_{t \to \infty} \frac{\int_t^{t+\omega} b_i(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{ik}}{\int_t^{t+\omega} b_i(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{ik}} > \limsup_{t \to \infty} \frac{a_i(t)}{a_{ij}(t)}, \quad \forall j \leq l, \tag{3.3}
\]

then species \( x_i \) \( (i = r + 1, r + 2, \ldots, n) \) are extinction, that is, for any positive solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1.2),

\[
\lim_{t \to \infty} x_i(t) = 0, \quad i = r + 1, r + 2, \ldots, n. \tag{3.4}
\]

Proof. Firstly, from assumption H, that (2.7) still holds and there are constants \( \eta_0 > 0 \) and \( T_0 > 0 \) such that

\[
\int_t^{t+\omega} b_i(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{ik} \geq \eta_0, \tag{3.5}
\]

for all \( t \geq T_0 \) and \( i = 1, 2, \ldots, n \).

We first prove \( x_n(t) \to 0 \) as \( t \to \infty \). Without loss of generality, we assume that condition (3.2) holds. When condition (3.3) holds, a similar argument can be given. Since

\[
\limsup_{t \to \infty} \frac{\int_t^{t+\omega} b_n(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{nk}}{\int_t^{t+\omega} b_p(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{pk}} < \liminf_{t \to \infty} \frac{a_n(t)}{a_{pj}(t)}, \quad j = 1, 2, \ldots, n, \tag{3.6}
\]

where \( p = i_n \). Hence, we can choose positive constants \( \alpha, \beta, \epsilon \) and \( T_n \geq T_0 \) such that

\[
\frac{\int_t^{t+\omega} b_n(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{nk}}{\int_t^{t+\omega} b_p(s)ds + \sum_{l \leq l_k < t+\omega} \ln h_{pk}} < \frac{\alpha}{\beta} - \epsilon < \frac{\alpha}{\beta} < \frac{a_n(t)}{a_{pj}(t)}, \tag{3.7}
\]
for all \( t \geq T_n \) and \( j = 1, 2, \ldots, n \). Hence, from (3.5) we further obtain

\[
\int_t^{t+\omega} \left( -ab_p(s) + \beta b_n(s) \right) ds + \beta \sum_{t \leq s < t+\omega} \ln h_{nk} - \alpha \sum_{t \leq s < t+\omega} \ln h_{pk} < -\beta \varepsilon \left( \int_t^{t+\omega} b_p(s) ds + \sum_{t \leq s < t+\omega} \ln h_{pk} \right)
\]

\[
\leq -\beta \varepsilon \eta_0,
\]

\[
\alpha a_{pj}(t) - \beta a_{nj}(t) = \beta a_{pj} \left[ \frac{\alpha}{\beta} \frac{a_{nj}(t)}{a_{pj}(t)} \right] < 0,
\]

for all \( t \geq T_n \) and \( j = 1, 2, \ldots, n \).

Consider the Liapunov function as follows:

\[
V_n(t) = \left( x_p(t) \right)^{-\alpha} (x_n(t))^\beta.
\]

Calculating the derivative, and from (3.1), we can obtain for any \( t \geq 0 \)

\[
\frac{dV_n(t)}{dt} = V_n(t) \left[ -\alpha \left( b_p(t) - \sum_{j=1}^{n} a_{pj}(t) x_j^{\theta_{pj}}(t) \right) + \beta \left( b_n(t) - \sum_{j=1}^{n} a_{nj}(t) x_j^{\theta_{nj}}(t) \right) \right]
\]

\[
= V_n(t) \left[ -\alpha b_p(t) + \beta b_n(t) + \sum_{j=1}^{n} \left( \alpha a_{pj}(t) - \beta a_{nj}(t) \right) x_j^{\theta_{pj}}(t) \right],
\]

for all \( t \neq t_k \) and

\[
V_n(t_k^+) = h_{pk}^{-\alpha} h_{nk}^\beta V_n(t_k),
\]

for all \( k = 1, 2, \ldots \). From (3.9), we further have

\[
\frac{dV_n(t)}{dt} \leq V_n(t) \left( -\alpha b_p(t) + \beta b_n(t) \right), \quad t \geq T_n, \quad t \neq t_k,
\]

\[
V_n(t_k^+) = h_{pk}^{-\alpha} h_{nk}^\beta V_n(t_k), \quad k = 1, 2, \ldots.
\]
For any $t > T_n$, there is an integer $q_t \geq 0$ such that $t \in [T_n + q_t \omega, T_n + (q_t + 1)\omega]$. Hence, by integrating (3.13) from $T_n$ to $t$, we obtain

$$V_n(t) \leq V_n(T_n) \exp\left(\int_{T_n}^{t} [-a\bar{b}_p(s) + \beta\bar{b}_n(s)]ds + \sum_{T_n \leq s \leq t} \ln\left(h_{pk}^{-\alpha} h_{nk}^{\beta}\right)\right)$$

$$= V_n(T_n) \exp\left\{\int_{T_n}^{t} [-a\bar{b}_p(s) + \beta\bar{b}_n(s)]ds + \sum_{T_n \leq s \leq t} \ln\left(h_{pk}^{-\alpha} h_{nk}^{\beta}\right)\right\}$$

$$\leq M_n \exp(-\varepsilon \beta \eta_0 q_t),$$

where

$$M_n = V_n(T_n) \exp\left(\omega \sup_{t \geq 0} \{a|\bar{b}_p(t)| + \beta|\bar{b}_n(t)|\} + (\alpha + \beta)P\right).$$

Since $q_t \to \infty$ as $t \to \infty$, it follows that from (3.14)

$$V_n(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (3.16)$$

Since

$$(x_n(t))^{\beta} = V_n(t) (x_{p}(t))^{\alpha},$$

$$(h_{nk}x_n(t_k))^{\beta} = h_{pk}^{-\alpha} x_{p}(t_k) h_{nk}^{\beta} V_n(t_k),$$

by the boundedness of $x(t)$ on $[0, \infty)$ (see Lemma 2.2), we have

$$x_n(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (3.18)$$
For any integer \( l > r \), assume that we have obtained \( x_i(t) \to 0 \) as \( t \to \infty \) for all \( i > l \). Now, we prove that \( x_i(t) \to 0 \) as \( t \to \infty \). Suppose that condition (3.3) holds. When condition (3.2) holds, the argument is similar. Let \( i = i_l \), by (3.1), we have \( \theta_{i_l} = \theta_{l_l} \), then for \( j \leq l \), we have \( \theta_{l_j} = \theta_{l_j} \). Then we can choose positive constants \( \lambda, \eta, \delta \) and \( T_l \geq T_0 \) such that

\[
\int_t^{t+\omega} b_i(s)ds + \sum_{t \leq s < t+\omega} \ln h_{ik} > \frac{\lambda}{\eta} + \delta > \frac{\lambda}{\eta} \frac{a_{ij}(t)}{a_{il}(t)}, \tag{3.19}
\]

for all \( t \geq T_l \) and \( j = 1, 2, \ldots, l \), where \( q = i_l \).

Consider the Liapunov function as follows:

\[
V_l(t) = (x_q(t))^{-\eta}(x_l(t))^\lambda. \tag{3.20}
\]

By calculating, we obtain for any \( t \geq 0 \)

\[
\frac{dV_l(t)}{dt} = V_l(t) \left[ -\eta b_q(t) + \lambda b_l(t) + \sum_{j=1}^l (\eta a_{lj}(t) - \lambda a_{lj}(t)) x_j^{\theta_{lj}}(t) \right. \\
+ \sum_{j=l+1}^n \eta a_{lj}(t) x_j^{\theta_{lj}}(t) - \sum_{j=l+1}^n \lambda a_{lj}(t) x_j^{\theta_{lj}}(t) \left. \right] , \tag{3.21}
\]

for all \( t \neq t_k \) and

\[
V_l(t_k^+) = h_{qk}^{-\eta} h_{lk}^\lambda V_l(t_k), \tag{3.22}
\]

for all \( k = 1, 2, \ldots \). From (3.3) and (3.19), we have

\[
\int_t^{t+\omega} (-\eta b_q(s) + \lambda b_l(s))ds + \lambda \sum_{t \leq s < t+\omega} \ln h_{lk} - \eta \sum_{t \leq s < t+\omega} \ln h_{qk} \\
< -\eta \delta \left( \int_t^{t+\omega} b_l(s)ds + \sum_{t \leq s < t+\omega} \ln h_{ik} \right) \tag{3.23}
\]

\[
\leq -\delta \eta \eta_0, \eta a_{lj}(t) - \lambda a_{lj}(t) < 0, \tag{3.24}
\]
for all \( t \geq T_l \) and \( j = 1, 2, \ldots, l \). Hence, from (3.21), it follows that

\[
\frac{dV_l(t)}{dt} \leq V_l(t) \left[ -\eta b_l(t) + \lambda b_l(t) + \sum_{j=1}^{l} (\eta a_{l j}(t) - \lambda a_{l j}(t)) x_j^{\theta_j}(t) + \sum_{j=l+1}^{n} \eta a_{l j}(t) x_j^{\theta_j}(t) - \sum_{j=l+1}^{n} \lambda a_{l j}(t) x_j^{\theta_j}(t) \right], \quad t \geq T_l, \; t \neq t_k,
\]

(3.25)

\[ V_n(t^+_k) = h_{pk}^\beta h_{nk}^\beta V_n(t_k), \quad k = 1, 2, \ldots. \]

Since \( x_i(t) \to 0 \) as \( t \to \infty \) for all \( i > l \), by the boundedness of \( a_{ij}(t) \) \( (i, j = 1, 2, \ldots, n) \) on \( [0, \infty) \), we obtain

\[
\lim_{t \to \infty} \int_t^{t+\omega} \sum_{j=l+1}^{n} \left( \eta a_{l j}(s) x_j^{\theta_j}(s) - \lambda a_{l j}(s) \right) x_j^{\theta_j}(s)ds = 0. \tag{3.26}
\]

Hence, for any small \( \varepsilon > 0 \), there is a \( T'_l > 0 \), such that

\[
\int_t^{t+\omega} \sum_{j=l+1}^{n} \left( \eta a_{l j}(s) x_j^{\theta_j}(s) - \lambda a_{l j}(s) \right) x_j^{\theta_j}(s)ds < \varepsilon, \quad t > T'_l. \tag{3.27}
\]

Combining (3.23), it follows that there is enough large \( T^*_l > \max\{T_l, T'_l\} \) such that for all \( t \geq T^*_l \),

\[
\int_t^{t+\omega} \left[ -\eta b_l(s) + \lambda b_l(s) + \sum_{j=l+1}^{n} \left( \eta a_{l j}(s) x_j^{\theta_j}(s) - \sum_{j=l+1}^{n} \lambda a_{l j}(s) \right) x_j^{\theta_j}(s) \right] ds
\]

\[
-\eta \sum_{t \leq t_k < t+\omega} \ln h_{ik} + \lambda \sum_{t \leq t_k < t+\omega} \ln h_{ik} \leq -\frac{1}{2} \delta \eta \delta, \tag{3.28}
\]

\[ x_i(t) \leq \delta \quad \forall i > l. \]
For any $t > T^*_i$, we firstly choose an integer $q_i \geq 0$ such that $t \in (T^*_i + q_i \omega, T^*_i + (q_i + 1)\omega]$. Integrating (3.25) from $T^*_i$ to $t$, then from (3.3) and (3.28), we have

$$V_i(t) \leq V_i(T^*_i) \exp \left\{ \int_{T^*_i}^{t} \left[ -\eta b_q(s) + \lambda b_l(s) + \sum_{j=1}^{n} \eta a_{qj}(s) x_j^{\theta_j}(s) - \sum_{j=1}^{n} \lambda a_{lj}(s) x_j^{\theta_j}(s) \right] ds + \sum_{T^*_i \leq s < t} \ln \left( h^{-\eta}_{q_k} h_k \right) \right\}$$

$$= V_i(T^*_i) \exp \left\{ \left( \int_{T^*_i}^{T^*_i + \omega} + \sum_{T^*_i \leq s < T^*_i + \omega} \ln \left( h^{-\eta}_{q_k} h_k \right) \right) + \cdots \right\}$$

$$+ \left( \int_{T^*_i + q_i \omega}^{T^*_i + (q_i + 1) \omega} \left[ -\eta b_q(s) + \lambda b_l(s) + \sum_{j=1}^{n} \eta a_{qj}(s) x_j^{\theta_j}(s) - \sum_{j=1}^{n} \lambda a_{lj}(s) x_j^{\theta_j}(s) \right] ds + \sum_{T^*_i + (q_i + 1) \omega \leq s < T^*_i + q_i \omega} \ln \left( h^{-\eta}_{q_k} h_k \right) \right)$$

$$+ \left( \int_{T^*_i + q_i \omega}^{t} \left[ -\eta b_q(s) + \lambda b_l(s) + \sum_{j=1}^{n} \eta a_{qj}(s) x_j^{\theta_j}(s) - \sum_{j=1}^{n} \lambda a_{lj}(s) x_j^{\theta_j}(s) \right] ds + \sum_{T^*_i + q_i \omega \leq s < t} \ln \left( h^{-\eta}_{q_k} h_k \right) \right) \right\}$$

$$\leq M_i \exp \left( -\frac{1}{2} \delta \eta_0 q_i \right).$$

(3.29)

where

$$M_i = V_i(T^*_i) \exp \left\{ \omega \sup_{t \geq 0} \left\{ \eta |b_q(t)| + \lambda |b_l(t)| + \sum_{j=1}^{n} (\eta a_{qj}(t) + \lambda a_{lj}(t)) \theta_j \right\} + (\lambda + \eta) P \right\}.$$  

(3.30)

Since $q_i \to \infty$ as $t \to \infty$, we obtain from (3.29)

$$V_i(t) \to 0 \quad \text{as} \quad t \to \infty,$$  

(3.31)
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Since

\[ (x_i(t))^\eta = V_i(t)(x_q(t))^\lambda, \]
\[ (h_{ik}x_i(t_k))^\eta = h_{ik}^\eta h_{qk}^{-\lambda} V_i(t_k)(h_{qk}x_q(t_k))^\lambda, \]  (3.32)

by the boundedness of \( x(t) \) on \([0, \infty)\), it follows that

\[ x_i(t) \rightarrow 0 \text{ as } t \rightarrow \infty. \]  (3.33)

Finally, by the induction principle, we obtain that \( x_i(t) \rightarrow 0 \) as \( t \rightarrow \infty \) for all \( i > r \). This completes the proof of Theorem 3.1.

4. Permanence

In this section, we study the permanence of partial species \( x_i(t) \) \((i = 1, 2, \ldots, r)\) of system (1.2). We state and prove the following result.

**Theorem 4.1.** Suppose that all the conditions of Theorem 3.1 hold. If for each \( i = 1, 2, \ldots, r \)

\[
\liminf_{t \to \infty} \left( \int_t^{t+\omega} \left[ b_i(s) - \sum_{j \neq i} a_{ij}(s) u_{i0}(t) \right] ds + \sum_{k \geq 1} \ln h_{ik} \right) > 0,
\]  (4.1)

where \( u_{i0} \) is some fixed positive solution of (2.3), then species \( x_i \) \((i = 1, 2, \ldots, r)\) are permanent, that is, there are positive constants \( m \) and \( M \) such that for any positive solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1.2)

\[ m \leq \liminf_{t \to \infty} x_i(t) \leq \limsup_{t \to \infty} x_i(t) \leq M, \quad i = 1, 2, \ldots, r. \]  (4.2)
Proof. From (4.1) and the boundedness of functions $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ on $R_+$, there are constants $\varepsilon_0 > 0$ and $T_1 > 0$ such that for any $t \geq T_1$ and $i = 1, 2, \ldots, r$. 

$$\int_t^{t+\omega} \left[ b_i(s) - \sum_{j=1}^n a_{ij}(s)\varepsilon_0 - \sum_{j \neq i}^r a_{ij}(s)u_{ij}^b(s) \right] ds + \sum_{t \leq s < t+\omega} \ln h_{ik} > \varepsilon_0. \quad (4.3)$$

For any $i \leq r$, from system (1.2), we have

$$\frac{dx_i(t)}{dt} \leq x_i(t) \left[ b_i(t) - \sum_{j=1}^n a_{ij}(t)x_j^\theta_i(t) \right],$$

$$\leq x_i(t) \left[ a_i(t) - b_{ii}(t)x_i^\theta_i(t) \right], \quad t \neq t_k, \quad t \geq 0,$$

$$x_i(t_k^+) = h_{ik}x_i(t_k), \quad k = 1, 2, \ldots,$$

we have

$$x_i(t) \leq u_i(t) \quad \forall t \geq 0, \quad (4.5)$$

where $u_i(t)$ is the solution of (2.3) with initial condition $u_i(0) \geq x_i(0)$. From Lemma 2.1 and Theorem 3.1, for the above constant $\varepsilon_0$ there is a $T_2 \geq T_1$ such that for all $t \geq T_2$

$$x_i(t) \leq u_i(t) \leq u_{0i}(t) + \varepsilon_0, \quad i = 1, 2, \ldots, r, \quad (4.6)$$

$$x_i(t) < \varepsilon_0, \quad i = r + 1, r + 2, \ldots, n. \quad (4.7)$$

Let

$$\gamma_i = \sup_{t \geq 0} \left\{ |b_i(t)| + \sum_{j=1}^n a_{ij}(t)\varepsilon_0 + \sum_{j \neq i}^r a_{ij}(t)u_{ij}^b(t) \right\},$$

$$m = \min_{1 \leq s \leq r} \{ \varepsilon_0 \exp(-\gamma_i\omega - P) \},$$

where constant $P > 0$ is given in (2.7). Obviously, $m > 0$ and $m$ is independent of any positive solution of system (1.2).

Now, we prove that there is a $T_3 \geq T_2$ such that

$$x_i(t) \geq m \quad \forall t \geq T_3, \quad i = 1, 2, \ldots, r. \quad (4.9)$$
We only need to consider the following three cases for each $i = 1, 2, \ldots, r$.

**Case I.** There is a $t_1 \geq T_2$ such that $x_i(t) \leq \epsilon'_0 = \sqrt{\epsilon_0}$ for all $t \geq t_1$.

**Case II.** There is a $t_2 \geq T_2$ such that $x_i(t) \geq \epsilon'_0$ for all $t \geq t_2$.

**Case III.** $x_i(t)$ oscillates about $\epsilon'_0$ for all $t \geq T_2$.

For Case I, let $t = t_1 + l\omega$, where $l \geq 0$ is any integer. From (4.3)–(4.7) we obtain

$$x_i(t) = x_i(t_1) \exp \left( \int_{t_1}^{t_1 + \omega} \left( b_i(s) - a_{il}(s)x_i^{\theta_i}(s) - \sum_{j \neq i} a_{ij}(s)x_j^{\theta_j}(s) \right) ds + \sum_{t_1 \leq \omega < t} \ln h_{ik} \right)$$

$$\geq x_i(t_1) \exp \left( \int_{t_1}^{t_1 + \omega} \left( b_i(s) - \sum_{j=1}^{n} a_{ij}(s)\epsilon_0 - \sum_{j \neq i}^{r} a_{ij}(s)u_j^{\theta_j}(s) \right) ds \right)$$

$$+ \sum_{t_1 \leq \omega < t_1 + \omega} \ln h_{ik} + \ldots + \int_{t_1 + (l-1)\omega}^{t_1 + l\omega} \left( b_i(s) - \sum_{j=1}^{n} a_{ij}(s)\epsilon_0 - \sum_{j \neq i}^{r} a_{ij}(s)u_j^{\theta_j}(s) \right) ds$$

$$+ \sum_{t_1 + (l-1)\omega \leq \omega < t_1 + l\omega} \ln h_{ik} \right)$$

$$\geq x_i(t_1) \exp(l\epsilon_0).$$

(4.10)

Therefore, $x_i(t) \to \infty$ as $l \to \infty$ which leads to a contradiction.

For Case III, we choose two sequences $\{\rho_n\}$ and $\{\rho^*_n\}$ satisfying $T_2 \leq \rho_1 < \rho^*_1 < \cdots < \rho_n < \rho^*_n < \cdots$ and $\lim_{n \to \infty} \rho_n = \lim_{n \to \infty} \rho^*_n = \infty$ such that

$$x_i(\rho_n) \geq \epsilon'_0, \quad x_i(\rho^*_n) \leq \epsilon'_0, \quad x_i(\rho^*_n) \leq \epsilon'_0, \quad x_i(\rho^*_n) \geq \epsilon'_0,$$

$$x_i(t) \leq \epsilon'_0 \quad \forall t \in (\rho_n, \rho^*_n),$$

$$x_i(t) \geq \epsilon'_0 \quad \forall t \in (\rho^*_n, \rho^*_{n+1}).$$

(4.11)

For any $t \geq T_2$, if $t \in (\rho_n, \rho^*_n]$ for some integer $n$, then we can choose an integer $l \geq 0$ such that $t = \rho_n + l\omega + \nu_i$, where $\nu_i \in [0, \omega)$ is a constant. Since for any $t \in (\rho_n, \rho^*_n)$ from (4.6) and (4.7) we have

$$\dot{x}_i(t) \geq x_i(t) \left( b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\epsilon_0 - \sum_{j \neq i}^{r} b_{ij}(t)u_j^{\theta_j}(t) \right), \quad t \neq t_k.$$  

(4.12)
Integrating this inequality from \( \rho_n \) to \( t \), then from (4.7) and (3.28)-(3.29) we have

\[
x_i(t) \geq x(\rho_n) \exp \left( \int_{\rho_n}^{t} \left( b_i(s) - \sum_{j=1}^{n} a_{ij}(s) \varepsilon_0 - \sum_{j \neq i}^{r} a_{ij}(s) u_{j0}^{\theta_j}(s) \right) ds + \sum_{\rho_n \leq t < \infty} \ln h_{ik} \right)
\]

\[
\geq \varepsilon_0 \exp \left( \int_{\rho_n}^{\rho_n + \omega} \left( b_i(s) - \sum_{j=1}^{n} a_{ij}(s) \varepsilon_0 - \sum_{j \neq i}^{r} a_{ij}(s) u_{j0}^{\theta_j}(s) \right) ds \right.
\]

\[
+ \sum_{\rho_n \leq t < \rho_n + \omega} \ln h_{ik} \left. \right) + \int_{\rho_n + \omega}^{\rho_n + \omega + \nu_i} \left( b_i(s) - \sum_{j=1}^{n} a_{ij}(s) \varepsilon_0 - \sum_{j \neq i}^{r} a_{ij}(s) u_{j0}^{\theta_j}(s) \right) ds + \sum_{\rho_n + \omega + \nu_i \leq t < \rho_n + \omega + \nu_i} \ln h_{ik}
\]

\[
\geq \varepsilon_0 \exp (-\gamma \omega - P).
\]

(4.13)

If there exists an integer \( n \) such that \( t \in (\rho_n^*, \rho_n + 1) \), then we obviously have

\[
x_i(t) \geq \varepsilon_0 > \varepsilon_0 \exp (-\gamma \omega - P).
\]

(4.14)

This shows that for Case III we always have

\[
x_i(t) \geq \varepsilon_0 \exp (-\gamma \omega - P), \quad \forall t \geq T_2.
\]

(4.15)

Finally, if Case II holds, then from \( x_i(t) \geq \varepsilon'_0 \) for all \( t \geq t_1 \), we can directly obtain that (4.9) holds.

Therefore, from Lemma 2.2 and (4.9), it follows that species \( x_i(t) (i = 1, 2, \ldots, r) \) are permanent. This proof of Theorem 4.1 is completed.

\[
5. \text{Global Attractivity}
\]

In this section, we further discuss the global attractivity of species \( x_i(t) (i \leq r) \). In order to obtain our results, we first consider the following subsystem which is composed of the species
\[ \frac{du_i(t)}{dt} = u_i(t) \left[ b_i(t) - \sum_{j=1}^{r} a_{ij}(t) u_j^{\theta_{ij}}(t) \right], \quad t \neq t_k, \quad (5.1) \]

\[ u_i(t_k^+) = h_{ik}(t_k), \quad i = 1, 2, \ldots, r, \quad k = 1, 2, \ldots. \]

We need the following lemma.

**Lemma 5.1.** Suppose that assumption \( H \) and condition (4.1) of Theorem 4.1 hold. Then subsystem (5.1) is permanent.

Lemma 5.1 can be proved by using the same method given in the proof of Theorem 4.1. We now state and prove the main result of this section.

**Theorem 5.2.** Suppose that all conditions of Theorem 3.1 and Theorem 4.1 hold. If there are positive constants \( \rho, D \) and \( d_i \) \( (i = 1, 2, \ldots, r) \) and nonnegative integrable function \( \mu(t) \) defined on \( \mathbb{R}_+ \), satisfying \( \int_{s}^{t} \mu(\tau) d\tau \geq -D + \rho(t - s) \) for all \( t \geq s \geq 0 \), such that

\[ d_i a_i(t) - \sum_{j \neq i} d_j a_{ji}(t) \geq \mu(t), \quad i = 1, 2, \ldots, r, \quad (5.2) \]

for all \( t \geq 0 \), then for any positive solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1.2) and any positive solution \( u(t) = (u_1(t), u_2(t), \ldots, u_r(t)) \) of subsystem (5.1)

\[ \lim_{t \to \infty} (x_i(t) - u_i(t)) = 0, \quad i = 1, 2, \ldots, r. \quad (5.3) \]

**Proof.** Let \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) be a positive solution of system (1.2) and \( u(t) = (u_1(t), u_2(t), \ldots, u_r(t)) \) be a positive solution of subsystem (5.1). By Theorem 3.1, we have \( x_i(t) \to 0 \) as \( t \to \infty \) for all \( i > r \). From Theorem 4.1 and Lemma 5.1, there are positive constants \( m \) and \( M \) such that

\[ m \leq x_i(t), \quad u_i(t) \leq M, \quad i = 1, 2, \ldots, r, \quad (5.4) \]

for all \( t \geq 0 \). Choose the Liapunov function as follows:

\[ V_r(t) = \sum_{i=1}^{r} d_i |\ln x_i(t) - \ln u_i(t)|. \quad (5.5) \]
Since
\[
V_r(t_k) = \sum_{i=1}^{r} d_i |\ln x_i(t_k) - \ln u_i(t_k)|
\]
\[
= \sum_{i=1}^{r} d_i |\ln h_{ik} x_i(t_k) - \ln h_{ik} u_i(t_k)|
\]
\[
= V_r(t_k),
\]
then \( V(t) \) is continuous for all \( t \geq 0 \). Calculating the upper right derivative of \( V_r(t) \), we have
\[
D^+ V_r(t) \leq \sum_{i=1}^{r} d_i \left( -a_{ii} \left| x_i^{\theta_i}(t) - u_i^{\theta_i}(t) \right| + \sum_{j \neq i}^{r} a_{ij}(t) \left| x_j^{\theta_i}(t) - u_j^{\theta_i}(t) \right| \right) + g(t) 
\]
\[
= -\sum_{i=1}^{r} \left( d_i a_{ii} - \sum_{j \neq i}^{r} d_j a_{ji}(t) \right) \left| x_i^{\theta_i}(t) - u_i^{\theta_i}(t) \right| + g(t),
\]
for all \( t \geq 0 \), where
\[
g(t) = \sum_{i=1}^{r} d_i \sum_{j=r+1}^{n} a_{ji}(t) x_j^{\theta_i}(t).
\]
By (5.2), we have
\[
D^+ V_r(t) \leq -\mu(t) \sum_{i=1}^{r} \left| x_i^{\theta_i}(t) - u_i^{\theta_i}(t) \right| + g(t), \quad \forall t \geq 0.
\]
By (5.4), we further obtain
\[
D^+ V_r(t) \leq -\lambda \mu(t) V_r(t) + g(t), \quad \forall t \geq 0,
\]
where \( \lambda = \min_{1 \leq i \leq r} d_i^{-1} m > 0 \). Applying the comparison theorem and the variation of constants formula of first-order linear differential equation, we have
\[
V_r(t) \leq e^{-\int_{0}^{t} \lambda \mu(s) ds} \left( \int_{0}^{t} g(s) e^{\int_{0}^{\tau} \lambda \mu(r) dr} d\tau + V_r(0) \right),
\]
for all \( t \geq 0 \). Since \( g(t) \to 0 \) as \( t \to \infty \), from the properties of function \( \mu(t) \) and (5.11), it is not hard to obtain \( V_r(t) \to 0 \) as \( t \to \infty \). That shows
\[
\lim_{t \to \infty} (x_i(t) - u_i(t)) = 0, \quad i = 1, 2, \ldots, r.
\]
This completes the proof of Theorem 5.2.
References


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