Research Article
Global Attractivity of a Family of Max-Type Difference Equations

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We propose to study a generalized family of max-type difference equations and then prove the global attractivity of a particular case of it under some parameter conditions. Through some numerical results of other cases, we finally pose a generic conjecture.

1. Introduction

The study of max-type difference equations is a hotspot in the area of discrete dynamics because such equations are often closely related to automatic control theory and competitive dynamics. For recent advances in this direction see [1–8] and the references therein.

Motivated by [9], Liu et al. [10] studied the following nonautonomous max-type difference equation:

\[ y_n = \frac{p + r y_{n-s}}{q + \phi_n(y_{n-1}, \ldots, y_{n-m}) + y_{n-s}}, \quad n \in \mathbb{N}_0, \tag{1.1} \]

where \( p \geq 0, r, q > 0, s, m \in \mathbb{N}, \) and \( \phi_n : (\mathbb{R}^+)^m \rightarrow \mathbb{R}^+, \) \( n \in \mathbb{N}_0 \) are mappings satisfying the condition \( \beta \min\{x_1, \ldots, x_m\} \leq \phi_n(x_1, x_2, \ldots, x_m) \leq \beta \max\{x_1, \ldots, x_m\}, \) for some fixed \( \beta \in (0, +\infty). \) When \( p = 0, \beta \in (0, 1), \) they proved that every positive solution to (1.1) converges to zero if \( r \leq q, \) while \( (r - q)/(1 + \beta) \) if \( r > q. \) If \( p > 0 \) and \( rq \geq p, \) then each positive solution to (1.1) converges to \( (\sqrt{(q - r)^2 + 4p(1 + \beta) - (q - r)}/(2(1 + \beta)), \) for some \( \beta \in (0, +\infty), \) except for the case \( q < r, \beta \in (\beta_0, +\infty), \) where \( \beta_0 = 4p/(q - r)^2 + 1. \) Note that the behavior of positive solutions to (1.1) for the case \( q < r, \beta \in (\beta_0, +\infty), \) is still an unsolved open problem as was mentioned in [10].
Here, we propose to investigate the asymptotic behavior of positive solutions to the generalized family of max-type difference equations

\[
x_n = \max_{1 \leq i \leq k} \left\{ \frac{p_i + r_i x_{n-s}}{q_i + x_{n-s} + f_i(x_{n-1}, \ldots, x_{n-m})} \right\}, \quad n \in \mathbb{N}_0, \tag{1.2}
\]

where \( p_i \geq 0, r_i, q_i > 0, s, m, k \in \mathbb{N}, k \geq 2 \) and the functions \( f_i : [0, +\infty)^m \to [0, +\infty), i = 1, 2, \ldots, k \) satisfy the condition

\[
\beta \min \{ u_1, \ldots, u_m \} \leq f_i(u_1, u_2, \ldots, u_m) \leq \beta \max \{ u_1, \ldots, u_m \}, \tag{1.3}
\]

for some fixed \( \beta \in (0, 1) \).

In this paper, we mainly consider the particular case that all \( p_i \) are zero, and then obviously (1.2) reduces to the following form:

\[
x_n = x_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + x_{n-s} + f_i(x_{n-1}, \ldots, x_{n-m})} \right\}, \quad n \in \mathbb{N}_0. \tag{1.4}
\]

Let \( x^* \) be a nonnegative equilibrium point of (1.4), then we have

\[
x^* = x^* \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (1 + \beta)x^*} \right\}. \tag{1.5}
\]

It follows directly from (1.5) that if \( 0 < r_i \leq q_i \) for all \( i = 1, 2, \ldots, k \), then (1.4) has the unique nonnegative equilibrium \( x^* = 0 \), while if there exists at least one \( j \in \{1, 2, \ldots, k\} \) such that \( r_j > q_j \), then (1.4) has a zero equilibrium \( x^* = 0 \) and a unique positive equilibrium \( x^* = \max_{1 \leq i \leq k} \{ r_i - q_i \}/(1 + \beta) \).

Finally, the following two beautiful theorems are derived.

**Theorem 1.1.** Consider (1.4) with condition (1.3). If \( 0 < r_i \leq q_i \) for all \( i = 1, 2, \ldots, k \), then every positive solution to (1.4) converges to the unique nonnegative equilibrium zero.

**Theorem 1.2.** Consider (1.4) with positive initial values and positive \( r_i \) and \( q_i \). Let \( f_i : [0, +\infty)^m \to [0, +\infty) \) be functions such that for some fixed \( \beta \in (0, 1) \), there hold

\[
\beta \min \{ u_1, \ldots, u_m \} \leq f_i(u_1, \ldots, u_m) \leq \beta \max \{ u_1, \ldots, u_m \}, \quad i = 1, 2, \ldots, k. \tag{1.6}
\]

If there exists at least one \( j \in \{1, 2, \ldots, k\} \) such that \( r_j > q_j \), then the unique positive equilibrium of (1.4) is a global attractor.

## 2. Preliminary Lemmas

For the purpose of establishing the main results, some auxiliary lemmas are essential.
Lemma 2.1. Consider the first-order difference equation

\[ x_n = x_{n-1} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + x_{n-1}} \right\}, \quad n \in \mathbb{N}_0, \tag{2.1} \]

with positive initial value \( x_{-1} \) and positive \( r_i \) and \( q_i \). If there exists at least one \( j \in \{1, 2, \ldots, k\} \) such that \( r_j > q_j \), then

\[ \lim_{n \to \infty} x_n = \max \{ r_i - q_i : i = 1, 2, \ldots, k \}. \tag{2.2} \]

Proof. Suppose that \( \max \{ r_i - q_i : i = 1, 2, \ldots, k \} = r_\tau - q_\tau \), which is positive, for some \( \tau \in \{1, 2, \ldots, k\} \). By making the variable change \( x_n = (r_\tau - q_\tau)y_n \) into (2.1) and then canceling the positive term \( r_\tau - q_\tau \) from the resulting equation, we can derive

\[ y_n = y_{n-1} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_\tau - q_\tau)y_{n-1}} \right\}, \quad n \in \mathbb{N}_0. \tag{2.3} \]

Note that \( \min\{a_i/b_1, a_2/b_2\} \leq (a_1 + a_2)/(b_1 + b_2) \leq \max\{a_i/b_1, a_2/b_2\} \) for \( a_i, b_i > 0, i = 1, 2 \). Then it follows from (2.3) that

\[ y_{n+1} = \max_{1 \leq i \leq k} \left\{ \frac{q_i y_n + (r_i - q_i)y_n}{q_i + (r_\tau - q_\tau)y_n} \right\} \leq \max_{1 \leq i \leq k} \left\{ \frac{q_i y_n + (r_\tau - q_\tau)y_n}{q_i + (r_\tau - q_\tau)y_n} \right\} \leq \max\{y_n, 1\}. \tag{2.4} \]

In addition, the following two inequalities hold:

\[ y_{n+1} - 1 = \max_{1 \leq i \leq k} \left\{ \frac{r_i y_n}{q_i + (r_\tau - q_\tau)y_n} - 1 \right\} \geq \frac{r_\tau y_n}{q_\tau + (r_\tau - q_\tau)y_n} - 1 = \frac{q_\tau (y_n - 1)}{q_\tau + (r_\tau - q_\tau)y_n}, \tag{2.5} \]

\[ y_{n+1} - y_n = \max_{1 \leq i \leq k} \left\{ \frac{r_i y_n}{q_i + (r_\tau - q_\tau)y_n} - y_n \right\} \geq \frac{r_\tau y_n}{q_\tau + (r_\tau - q_\tau)y_n} - y_n = \frac{(r_\tau - q_\tau)y_n(1 - y_n)}{q_\tau + (r_\tau - q_\tau)y_n}. \tag{2.6} \]

In the following, we are confronted with three possibilities.

Case 1. If there exists \( n_0 \geq -1 \) such that \( y_{n_0} = 1 \), then it follows from (2.4) and (2.5) that \( y_n = 1 \) holds for all \( n \geq n_0 \).

Case 2. If there exists \( n_0 \geq -1 \) such that \( y_{n_0} > 1 \), then it follows from (2.5) and (2.6) that

\[ y_{n_0} \geq y_{n_0+1} \geq y_{n_0+2} \geq \cdots > 1. \tag{2.7} \]
Thus there is a finite limit \( \gamma = \lim_{n \to \infty} y_n \geq 1 \). By taking the limits on both sides of (2.3) and canceling the positive factor \( \gamma \) from the resulting equation, we obtain

\[
1 = \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_{\tau} - q_{\tau}) \gamma} \right\},
\]

which implies \( \gamma = 1 \). Because if \( \gamma > 1 \), then

\[
1 = \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_{\tau} - q_{\tau}) \gamma} \right\} < \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_{\tau} - q_{\tau})} \right\} = 1,
\]

leading to a contradiction.

**Case 3.** If \( y_n < 1 \) for all \( n \geq -1 \), then it follows from (2.5) and (2.6) that

\[
y_{-1} < y_0 < y_1 < \ldots < y_n < \ldots < 1.
\]

Therefore, the limit of \( y_n \) exists, denoted by \( 0 < \gamma = \lim_{n \to \infty} y_n \leq 1 \). By taking the limits on both sides of (2.3) and canceling the nonzero factor \( \gamma \) from the resulting equation, there hold

\[
1 = \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_{\tau} - q_{\tau})} \right\},
\]

which implies \( \gamma = 1 \). Because if \( 0 < \gamma < 1 \), then

\[
1 = \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_{\tau} - q_{\tau})} \right\} > \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + (r_{\tau} - q_{\tau})} \right\} = 1,
\]

which is a contradiction.

In either of the above three cases, we get \( \lim_{n \to \infty} y_n = 1 \), implying \( \lim_{n \to \infty} x_n = r_{\tau} - q_{\tau} \).

From Lemma 2.1, we have the following result.

**Lemma 2.2.** Consider the \( s \)-order difference equation

\[
x_n = x_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + x_{n-s}} \right\}, \quad n \in \mathbb{N}_0,
\]

with positive initial values and \( r_i, q_i > 0 \). If there exists at least one \( j \in \{1, 2, \ldots, k\} \) such that \( r_j > q_j \), then

\[
\lim_{n \to \infty} x_n = \max\{r_i - q_i : i = 1, 2, \ldots, k\}.
\]
3. Proofs of Main Theorems

In this section, we are in a position to prove the main theorems presented in Section 1.

Proof of Theorem 1.1. Note that for the case \( r_i < q_i, \ i = 1, 2, \ldots, k \), the behavior of positive solutions to (1.4) is quite simple. In this case, we have that

\[
x_n \leq x_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i} \right\} = \mu x_{n-s},
\]  

(3.1)
where \( \mu = \max_{1 \leq i \leq k} \{ r_i/q_i \} < 1 \). Easily the subsequences \( \{ x_{l+s+j} \}_{l \in \mathbb{N}_0}, \; j \in \{0,1,\ldots,s-1\} \) converge to zero, hence the sequence \( \{ x_n \} \) also converges to zero.

For the case \( r_i \leq q_i, \; i = 1,2,\ldots,k \) with at least one \( j \in \{1,2,\ldots,k\} \) such that \( r_j = q_j \), we can obtain that

\[
    x_n \leq x_{n-s} \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i} \right\} = x_{n-s}.
\]  

(3.2)

In this case, the subsequences \( \{ x_{l+s+j} \}_{l \in \mathbb{N}_0}, \; j = 0,1,\ldots,s-1 \) are all positive and nonincreasing, thus they converge, respectively, to some nonnegative limits \( q_j := \lim_{l \to \infty} x_{l+s+j}, \; j = 0,1,\ldots,s-1 \).

If we replace \( n \) in (1.4) by \( sl+j, \; l \in \mathbb{N}_0 \) for an arbitrary fixed \( j \in \{0,1,\ldots,s-1\} \) and let \( l \to \infty \), we can get

\[
    q_j = q_j \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + q_j + f_i(q_{v_1},\ldots,q_{v_m})} \right\},
\]

(3.3)

where \( r_i \in \{0,1,\ldots,s-1\}, \; i = 1,\ldots,m \). Without loss of generality, assume that \( q_j \neq 0 \), then we obtain that

\[
    1 = \frac{r_\tau}{q_\tau + q_j + f_\tau(q_{v_1},\ldots,q_{v_m})},
\]

(3.4)

with some fixed number \( \tau \in \{1,2,\ldots,k\} \). Because \( r_\tau \leq q_\tau \), then it follows from (3.4) that

\[
    q_\tau + q_j + f_\tau(q_{v_1},\ldots,q_{v_m}) = r_\tau \leq q_\tau.
\]

(3.5)

Therefore we have

\[
    q_j + f_\tau(q_{v_1},\ldots,q_{v_m}) = 0,
\]

(3.6)

leading to \( q_j = 0 \), which is a contradiction. Hence we have that \( q_j = 0, \; j = 0,1,\ldots,s-1 \), and every positive solution to (1.4) converges to zero, if \( r_i \leq q_i \) for all \( i = 1,2,\ldots,k \). \( \Box \)

**Proof of Theorem 1.2.** Suppose that \( \max \{ r_i - q_i : i = 1,2,\ldots,k \} = r_\tau - q_\tau > 0 \) for some \( \tau \in \{1,2,\ldots,k\} \). Let \( \epsilon \) be an arbitrary fixed real number with \( 0 < \epsilon < (1-\beta)/(1+\beta)(r_\tau - q_\tau) \). Define two sequences \( \{ M_k \} \) and \( \{ m_k \} \) in the way shown in (2.15) with \( a = r_\tau, b = q_\tau \). \( \Box \)

Let \( \{ x_n \} \) be an arbitrary positive solution to (1.4). Next, we proceed by proving two claims.

**Claim 1.** There exists \( N_1 \in \mathbb{N} \) such that \( m_1 - \epsilon \leq x_n \leq M_1 + \epsilon \) for all \( n \geq N_1 \).

**Proof of Claim 1.** Note that

\[
    x_n \leq x_{n-s} \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + x_{n-s}} \right\}, \quad n = 0,1,2,\ldots.
\]

(3.7)
Consider the following difference equation:

$$z^{(1)}_n = z^{(1)}_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + z^{(1)}_{n-s}} \right\}, \quad n = 0, 1, 2, \ldots \tag{3.8}$$

Let \(\{z^{(1)}_n\}\) be a positive solution to (3.7) with the initial values \(z^{(1)}_{-1} = x_{-1}, z^{(1)}_{-2} = x_{-2}, \ldots, z^{(1)}_{-s} = x_{-s}\).

Note that the mapping \(h(x) = \frac{rx}{(q + x)}\) is strictly increasing on the interval \((0, +\infty)\).

It follows by induction that \(x_n \leq z^{(1)}_n\) for all \(n \geq -s\). By Lemma 2.2, we have \(\lim_{n \to \infty} z^{(1)}_n = r_r - q_r = M_1\). Hence there is an integer \(N_1' \in \mathbb{N}\) such that \(x_n \leq M_1 + \epsilon\) for \(n \geq N_1'\).

Let \(t = \max\{s, m\}\). Note that

$$x_n \geq x_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + x_{n-s} + \beta(M_1 + \epsilon)} \right\}, \quad n \geq N_1' + t. \tag{3.9}$$

Consider the difference equation

$$y^{(1)}_n = y^{(1)}_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + y^{(1)}_{n-s} + \beta(M_1 + \epsilon)} \right\}, \quad n \geq N_1' + t, \tag{3.10}$$

with \(y^{(1)}_{N_1'+t-1} = x_{N_1'+t-1}, y^{(1)}_{N_1'+t-2} = x_{N_1'+t-2}, \ldots, y^{(1)}_{N_1'} = x_{N_1'}\). Note the monotonicity of \(h(x)\), it follows by induction that \(x_n \geq y^{(1)}_n\) for all \(n \geq N_1'\). By Lemma 2.2, we get that \(\lim_{n \to \infty} y^{(1)}_n = m_1\). Thus there exists an integer \(N_1 \geq N_1'\) such that \(x_n \geq m_1 - \epsilon\) for all \(n \geq N_1\).

Working inductively, we will reach the following claim.

**Claim 2.** For every \(k \in \mathbb{N}\), there exists \(N_k \in \mathbb{N}\) such that

$$m_k - \frac{\epsilon}{K} \leq x_n \leq M_k + \frac{\epsilon}{K}, \tag{3.11}$$

for all \(n \geq N_k\).

**Proof of Claim 2.** Obviously, the case \(k = 1\) follows directly from Claim 1. In the following, we proceed by induction. Assume that the assertion is true for \(k = \omega(\omega \geq 1)\). Then it suffices to prove the assertion is also true for \(k = \omega + 1\).

Note that

$$x_n \leq x_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + x_{n-s} + \beta(m_\omega - \epsilon/\omega)} \right\}, \quad n \geq N_\omega + t. \tag{3.12}$$

Consider the difference equation

$$z^{(\omega+1)}_n = z^{(\omega+1)}_{n-s} \times \max_{1 \leq i \leq k} \left\{ \frac{r_i}{q_i + z^{(\omega+1)}_{n-s} + \beta(m_\omega - \epsilon/\omega)} \right\}, \quad n \geq N_\omega + t, \tag{3.13}$$
with $z^{(ω+1)}_{Nω+1} = x_{Nω+1−1}$, $z^{(ω+1)}_{Nω+1+1} = x_{Nω+1−2}$, ..., $z^{(ω+1)}_{Nω} = x_{Nω}$. Note the monotonicity of $h(x)$, it follows by induction that $x_n \leq z^{(ω+1)}_n$ for all $n \geq Nω$. By Lemma 2.2, we have that $\lim_{n \to \infty} z^{(ω+1)}_n = M_{ω+1}$. So there is an integer $N'_{ω+1} ∈ \mathbb{N}$ such that $x_n \leq M_{ω+1} + \epsilon/(ω + 1)$ for all $n \geq N'_{ω+1}$. Then note that

$$x_n \geq x_{n-s} \times \max_{1 \leq i \leq k}\frac{r_i}{q_i + x_{n-s} + \beta(M_{ω+1} + \epsilon/(ω + 1))}, \quad n \geq N'_{ω+1} + t. \quad (3.14)$$

Consider the following difference equation

$$y_n^{(ω+1)} = y_{n-s}^{(ω+1)} \times \max_{1 \leq i \leq k}\frac{r_i}{q_i + y_{n-s}^{(ω+1)} + \beta(M_{ω+1} + \epsilon/(ω + 1))}, \quad n \geq N'_{ω+1} + t, \quad (3.15)$$

with $y_{N'_{ω+1}+1}^{(ω+1)} = x_{N'_{ω+1}+1}$, $y_{N'_{ω+1}+2}^{(ω+1)} = x_{N'_{ω+1}+2}$, ..., $z^{(ω+1)}_{Nω} = x_{Nω}$. By the monotonicity of $h(x)$, it follows by induction that $x_n \geq y_n^{(ω+1)}$ for all $n \geq N'_{ω+1}$. By Lemma 2.2, we have that $\lim_{n \to \infty} y_n^{(ω+1)} = m_{ω+1}$. So there is an integer $N'_{ω+1} \geq N_{ω+1}$ such that $x_n \geq m_{ω+1} - \epsilon/(ω + 1)$ for all $n \geq N_{ω+1}$.

From Claim 2, we derive

$$\lim_{k \to \infty} m_k = \lim_{k \to \infty} \left( m_k - \frac{\epsilon}{k} \right) \leq \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n \leq \lim_{k \to \infty} \left( M_k + \frac{\epsilon}{k} \right) = \lim_{k \to \infty} M_k. \quad (3.16)$$

This plus Lemma 2.3 leads to that

$$\lim_{n \to \infty} x_n = \lim_{k \to \infty} m_k = \lim_{k \to \infty} M_k = \frac{r_T - q_T}{1 + \beta}. \quad (3.17)$$

4. Simulations and Future Work

In the previous section, we proved the global attractivity of (1.2) when all $p_i$ are zero. In this section, we investigate the dynamic behavior of (1.2) provided that all $p_i$ are not zero. First, it is trivial to confirm that when all $p_i$ are not zero, (1.2) has the following unique positive equilibrium point $x^* = \max_{1 \leq i \leq k} \{ \sqrt{(q_i - r_i)^2 + 4p_i(1 + \beta) + r_i - q_i} / (2(1 + \beta)) \}$. In the following, some numerical results are presented.

Experiment 1. Consider the first-order difference equation

$$x_n = \max \left\{ \frac{0.2 + 0.6x_{n-1}}{0.6 + x_{n-1} + 0.3x_{n-1}}, \frac{rx_{n-1}}{q + x_{n-1} + 0.3x_{n-1}} \right\}, \quad n \in \mathbb{N}, \quad (4.1)$$

where $r, q > 0$ and the initial value $x_0 > 0$. (See Figures 1 and 2).
Experiment 2. Consider the second-order difference equation

\[ x_n = \max \left\{ \frac{0.5 + x_{n-2}}{1 + x_{n-2} + 0.5x_{n-1}} \cdot \frac{0.8 + rx_{n-2}}{q + x_{n-2} + 0.5x_{n-1}} \right\}, \quad n \geq 2, \tag{4.2} \]

where \( r, q > 0 \) and the initial values \( x_0, x_1 > 0 \). (See Figures 3 and 4).

Experiment 3. Consider the third-order difference equation

\[ x_n = \max \left\{ \frac{0.5 + x_{n-3}}{1 + x_{n-3} + 0.9\sqrt{(x_{n-1}^2 + x_{n-2}^2)/2}} \cdot \frac{3x_{n-3}}{2 + x_{n-3} + 0.9\sqrt{(x_{n-1}^2 + x_{n-2}^2)/2}} \right\}, \quad n \geq 3, \tag{4.3} \]

where the initial values \( x_0, x_1, x_2 > 0 \). (See Figure 5).
Conjecture 4.1. Consider (1.2) with nonnegative $p_i$ and positive $r_i$ and $q_i$. Let $f_i : [0, +\infty)^m \to [0, +\infty)$, $i = 1, 2, \ldots, k$ be $k$ functions such that for some fixed $\beta \in (0, 1)$, there hold

$$\beta \min \{u_1, \ldots, u_k\} \leq f_i(u_1, \ldots, u_k) \leq \beta \max \{u_1, \ldots, u_k\}. \quad (4.4)$$

If $r_i q_i \geq p_i$ for all $i = 1, 2, \ldots, k$, then every positive solution to (1.2) converges to the equilibrium point

$$x^* = \frac{1}{2(1 + \beta)} \max_{1 \leq i \leq k} \left\{ \sqrt{(q_i - r_i)^2 + 4p_i(1 + \beta) + r_i - q_i} \right\}. \quad (4.5)$$
Figure 5: $x^* = 10/19 \approx 0.5263$.

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References


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